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Reversible polynomial automorphisms of the plane: the involutory case

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Abstract

Planar polynomial automorphisms are polynomial maps of the plane whose inverse is also a polynomial map. A map is reversible if it is conjugate to its inverse. Here we obtain a normal form for automorphisms that are reversible by an involution that is also in the group of polynomial automorphisms. This form is a composition of a sequence of generalized Hénon maps together with two simple involutions. We show that the coefficients in the normal form are unique up to finitely many choices.

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1. Introduction

A diffeomorphism g has a reversing symmetry, or is "reversible" if it is conjugate to its inverse [1–4], i.e., there exists a diffeomorphism R such that

$$g^{-1} = RgR^{-1}. \tag{1}$$

Reversible maps have a number of special properties and occur often in applications. For example, if the phase space of a system consists of configuration coordinates and momenta, then it is often the case that reversal of the momenta, $R(q,p) = (q,-p)$, corresponds to the reversal of time. Note that if R is a reversor, then it generates a family of reversors $g^n R$, for any integer n.

Reversibility is often associated with Hamiltonian systems, and more generally to conservative dynamics. Most representative examples of reversible systems originate in the study of Hamiltonian dynamics, and many Hamiltonian systems appearing in applications are reversible. It is, however, well known that these properties are independent: Hamiltonian systems need not be reversible, and reversible systems need not

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be conservative, or even volume-preserving [5] (see also [4] and references therein).

Often the reversor is an involution, $R^{-1} = R$. This is the case, for example, for the physical reversor mentioned above. When this is true, a reversible map can be written as the composition of two involutions

$$g = (gR)(R),$$

since gR is also an involution. For example, the standard [6] and area-preserving Hénon maps [2] have involutions as reversors.

Orbits of g that intersect the fixed sets of the family of reversors, $g^n R$, are called "symmetric orbits". Since $\mathrm{Fix}(R) = \{z\colon\ R(z) = z\}$, generally has lower dimension than the phase space, these orbits are special, and are easier to find than general orbits. Their bifurcations are also special; for example, in the two-dimensional case, pitchfork bifurcations are generic in one-parameter families of reversible maps [7]. From this point of view, a reversor that is an involution is particularly interesting, because its fixed set is often nontrivial. For example, every orientation-reversing involution of the plane has a one-dimensional fixed set [5].

Our goal is to classify reversible polynomial automorphisms of the plane. Polynomial maps form one of the simplest, nontrivial classes of nonlinear dynamical systems. A polynomial automorphism is a polynomial diffeomorphism whose inverse is also polynomial. Such maps can give rise to quite complicated dynamics as exemplified by the renowned Hénon quadratic map [8]. A larger family of maps consists of the generalized Hénon maps of the form

$$h : (x, y) \to \big(y, p(y) - \delta x\big), \tag{2}$$

where $\delta = \det(Dh) \neq 0$ is the Jacobian of the map and $p(y)$ is any polynomial of degree $\geqslant 2$. These maps are reversible when they are area and orientation-preserving, i.e., $\delta = 1$, or area-preserving and orientation-reversing, $\delta = -1$, providing that p is an even polynomial. Some of their dynamics has been studied in [9,10]. In [9] Friedland and Milnor prove that every polynomial automorphism that is not dynamically trivial is conjugate to a composition of generalized Hénon transformations.

A remarkable property of any polynomial automorphism is that the determinant of its Jacobian matrix is a nonzero constant (a famous unsolved question—the Jacobian conjecture—is to determine if all polynomial maps with a nonzero, constant Jacobian are polynomial automorphisms [11]). It follows as a simple consequence of (1) that when the Jacobians of g and R are constant, then g is volume-preserving. Therefore, polynomial automorphisms possessing polynomial reversors must be volume-preserving.

Since the composition of any two polynomial automorphisms is again a polynomial automorphism, the set of polynomial automorphisms of $\mathbb{R}^2$ or of the complex plane, $\mathbb{C}^2$, is a group, we call it $\mathcal{G}$. We will make use of the results of Jung on the structure of $\mathcal{G}$ to investigate polynomial automorphisms that are reversible. Our results also heavily use the normal form for elements of $\mathcal{G}$ in terms of generalized Hénon maps obtained by Friedland and Milnor [9].

For the purposes of this Letter, we restrict attention to the case that R is an involution. While this is a restriction, we will show in a forthcoming paper that noninvolutory reversors are exceptional in the sense that additional conditions on the generalized Hénon transformations are required [12]. It has been previously shown that all reversible polynomial mappings in generalized standard form possess involutory reversors [13].

We will classify all maps that are reversible by an involution in $\mathcal{G}$ and obtain unique normal forms for these maps under conjugacy in $\mathcal{G}$. Just as in [9], these normal forms involve compositions of generalized Hénon maps; however, in this case, the reversors are introduced by including two involutions in this composition. We will see that reversible maps are either dynamically trivial, or conjugate to a map of the form

$$\big(h_1^{-1} \cdots h_m^{-1}\big) r_1 (h_m \cdots h_1) r_0. \tag{3}$$

Here the maps r_i are involutions that can be constructed from "elementary" involutions (see Proposition 5) and the simple permutation

$$t : (x, y) \to (y, x). \tag{4}$$

Conversely, any map of the form (3) is reversible when the r_i are involutions.

Specifically our main result shows that there are three classes of reversible automorphisms:

Theorem 1. *Let g be a nontrivial reversible automorphism. Then g is conjugate to one of the following* *classes of maps*

(AA) $\left(h_1^{-1} \cdots h_m^{-1}\right) t \left(h_m \cdots h_1\right) t$;

(EA) $\left(h_1^{-1} \cdots h_m^{-1}\right) e_{m+1} \left(h_m \cdots h_1\right) t$;

(EE) $\left(t h_1^{-1} \cdots h_m^{-1}\right) e_{m+1} \left(h_m \cdots h_1 t\right) e_0$;

where h_i represents a Hénon transformation in the form (2) *and e_0, e_{m+1} are* elementary involutions, *i.e., maps of the form*

$$e : (x, y) \to \big(p(y) - \delta x, \epsilon y\big).$$

Here δ and $\epsilon \in \{-1, +1\}$ and $p(\epsilon y) = \delta p(y)$ (see Proposition 5). *Furthermore it can be required that the Hénon transformations $h_i = t e_i$, $i = 1, \ldots, m$, as well as the involutions e_0, e_{m+1} be normalized by choosing the leading coefficients of the polynomials p_i to be $+1$, and their centers of mass* (*the sum of their roots*) *to be* 0. *In this case the resulting composition is unique up to a finite number of choices.*

Here the terminology (**AA**), etc., refers to the cases that the two involutions are both affine or elementary or one of each. We prove this result in Section 4. Although Theorem 1 refers to polynomial automorphisms of the complex plane it is not difficult to specialize the result to the real case, as we stipulate next.

Remark. For real polynomial automorphisms, a slight modification of the arguments in the proof of Theorem 1 allows us to obtain real normal forms. To do this, we might need to allow the leading coefficient of one of the polynomials in the normal Hénon transformations to be -1 instead of 1. This occurs because, as we will see in Section 4, the equations that we need to solve to normalize the maps otherwise may not have real solutions.

2. Background

For future reference we include some basic terminology and results (see, e.g., [9] for more details).

We will study polynomial maps of the complex plane, $\mathbb{C}^2$, occasionally specializing to the real case. We let $\mathcal{G}$ denote the group of *polynomial automorphisms* of the complex plane, i.e., the set of bijective maps

$$g : (x, y) \to \big(X(x, y), Y(x, y)\big), \quad X, Y \in \mathbb{C}[x, y],$$

having a polynomial inverse. Here $\mathbb{C}[x, y]$ is the ring of polynomials in the variables x and y, with coefficients in $\mathbb{C}$. The *degree* of g is defined as the largest of the degrees of X and Y.

The subgroup $\mathcal{E} \subset \mathcal{G}$ of *elementary maps* consists of maps of the form

$$e : (x, y) \to \big(\alpha x + p(y), \beta y + \eta\big), \tag{5}$$

where $\alpha\beta \neq 0$ and $p(y)$ is any polynomial. The subgroup of affine automorphisms is denoted by $\mathcal{A}$. The affine maps that are also elementary will be denoted by $\mathcal{S} = \mathcal{A} \cap \mathcal{E}$, while $\widehat{\mathcal{S}}$ will denote the group of diagonal affine automorphisms,

$$\hat{s} : (x, y) \to (\alpha x + \xi, \beta y + \eta). \tag{6}$$

It is worthwhile to note that $\widehat{\mathcal{S}}$ is the largest subgroup of $\mathcal{S}$ normalized by $t : (x, y) \to (y, x)$, i.e., such that $t\widehat{\mathcal{S}}t = \widehat{\mathcal{S}}$. On the other hand a map $s \in \mathcal{S}$ commutes with t if it is a diagonal automorphism (6) with $\alpha = \beta$ and $\xi = \eta$. This subgroup of $\widehat{\mathcal{S}}$ is the centralizer of t in $\mathcal{S}$,

$$C_{\mathcal{S}}(t) = \big\{s \in \mathcal{S}\colon\ sts^{-1} = t\big\}.$$

Finally, conjugacy by t will be denoted by ϕ,

$$\phi(g) = tgt. \tag{7}$$

Thus if $s \in C_{\mathcal{S}}(t)$, then $\phi(s) = s$.

According to Jung's theorem [14] every polynomial automorphism $g \notin \mathcal{S}$, can be written as

$$g = g_m g_{m-1} \cdots g_2 g_1, \qquad g_i \in (\mathcal{E} \cup \mathcal{A}) \setminus \mathcal{S},\ i = 1, \ldots, m, \tag{8}$$

with consecutive terms belonging to different subgroups $\mathcal{A}$ or $\mathcal{E}$. An expression of the form (8) is called a *reduced word* of length m. An important property of a map written in this form is that its degree is the product of the degrees of the terms in the composition [9, Theorem 2.1]. As a consequence of this fact it can be seen that the identity cannot be expressed as a reduced word [9, Corollary 2.1]. This fact in turn means that $\mathcal{G}$ is the free product of $\mathcal{E}$ and $\mathcal{A}$ amalgamated along $\mathcal{S}$. The structure of $\mathcal{G}$ as an amalgamated free product determines the way in which reduced words corresponding to the same polynomial automorphism are related.

Theorem 2 (cf. [9, Corollary 2.3] or [15, Theorem 4.4]). *Two reduced words $g_m \cdots g_1$ and $\tilde{g}_n \cdots \tilde{g}_1$ represent the same polynomial automorphism g if and only if $n = m$ and there exist maps $s_i \in \mathcal{S}$, $i = 0, \ldots, m$ such that $s_0 = s_m = \mathrm{id}$ and $\tilde{g}_i = s_i g_i s_{i-1}^{-1}$.*

From this theorem it follows that the length of a reduced word (8) as well as the degrees of its terms are uniquely determined by g. The sequence of degrees $(l_1, \ldots, l_n)$ corresponding to the maps $(g_1, \ldots, g_m)$, after eliminating the 1's coming from affine terms, is called the *polydegree* of g.

A map is said to be *cyclically-reduced* in the trivial case that it belongs to $\mathcal{A} \cup \mathcal{E}$ or in the case that it can be written as a reduced word (8) with the additional conditions $m \geqslant 2$ and g_m, g_1 not in the same subgroup $\mathcal{E}$ or $\mathcal{A}$.

Two maps $g, \tilde{g} \in \mathcal{G}$ are conjugate in $\mathcal{G}$ if there exists $f \in \mathcal{G}$ such that $g = f\tilde{g}f^{-1}$. If f belongs to some subgroup $\mathcal{F}$ of $\mathcal{G}$ we say that g and $\tilde{g}$ are $\mathcal{F}$-conjugate. It can be seen that every $g \in \mathcal{G}$ is conjugate to a cyclically-reduced map. It can also be proved that every affine map a can be written as $a = st\tilde{s}$, with $t:(x, y) \to (y, x)$ and $s, \tilde{s}$ affine elementary maps. From these facts it follows that every polynomial automorphism that is not conjugate to an elementary or an affine map is conjugate to a reduced word of the form,

$$\begin{aligned} &g = te_m \cdots te_2 te_1, \\ &\quad e_i \in \mathcal{E} \setminus \mathcal{S},\ i = 1, \ldots, m,\ m \geqslant 1. \end{aligned} \tag{9}$$

Moreover this representative of the conjugacy class is unique up to modifications of the maps e_i by diagonal affine automorphisms and cyclic reordering. More precisely we have the following theorem (cf. [15, Theorem 4.6]).

Theorem 3. *Two nontrivial, cyclically-reduced words $g = g_m \cdots g_1$ and $\tilde{g} = \tilde{g}_n \cdots \tilde{g}_1$ are conjugate if and only if they have the same length and there exist automorphisms $s_i \in \mathcal{S}$, $i = 0, \ldots, m$, with $s_m = s_0$, and a cyclic permutation,*

$$(\hat{g}_m, \ldots, \hat{g}_1) = (\tilde{g}_k, \ldots, \tilde{g}_1, \tilde{g}_m, \ldots, \tilde{g}_{k+1})$$

such that $\hat{g}_i = s_i g_i s_{i-1}^{-1}$. In that case,

$$s_0 g s_0^{-1} = \hat{g}_m \cdots \hat{g}_1.$$

In particular, if $g = te_m \cdots te_1$ and $\tilde{g} = t\tilde{e}_m \cdots t\tilde{e}_1$ are conjugate, there exist diagonal automorphisms $s_i \in \widehat{\mathcal{S}}$, $s_m = s_0$, and a cyclic reordering,

$$(\hat{e}_m, \ldots, \hat{e}_1) = (\tilde{e}_k, \ldots, \tilde{e}_1, \tilde{e}_m, \ldots, \tilde{e}_{k+1}),$$

such that $t\hat{e}_i = s_i t e_i s_{i-1}^{-1}$ and

$$s_0 g s_0^{-1} = t\hat{e}_m \cdots t\hat{e}_1.$$

Proof. Let us consider $g = g_m \cdots g_1$ and $\tilde{g} = \tilde{g}_n \cdots \tilde{g}_1$, two nontrivial, cyclically-reduced, conjugate words. By assumption, there is a reduced word $f = f_k \cdots f_1 \in \mathcal{G}$, such that $g = f\tilde{g}f^{-1}$. Then,

$$g_m \cdots g_1 = f_k \cdots f_1 \tilde{g}_n \cdots \tilde{g}_1 f_1^{-1} \cdots f_k^{-1}. \tag{10}$$

However, the word on the right-hand side of (10) is not reduced. Since $\tilde{g}$ is cyclically-reduced, we can suppose, with no loss of generality, that f_1 and $\tilde{g}_n$ belong to the same subgroup $\mathcal{A}$ or E, so that f_1^{-1} and $\tilde{g}_1$ lie in different subgroups. Taking into account Theorem 2 and that (10) represents a cyclically-reduced map, we can reduce to obtain

$$f_k \cdots f_1 \tilde{g}_n \cdots \tilde{g}_1 = \begin{cases} s_k \tilde{g}_{n-k} \cdots \tilde{g}_1, & \text{if } n \geqslant k, \\ f_k \cdots f_{n+1} s_n, & \text{if } n < k, \end{cases} \tag{11}$$

where $s_n, s_k \in \mathcal{S}$. Moreover there exist $s_i \in \mathcal{S}$, $s_0 = \mathrm{id}$, such that $f_i s_{i-1} \tilde{g}_{n-i+1} = s_i$ for $i = 1, \ldots, \min(n, k)$.

For the case $n \geqslant k$,

$$\begin{aligned} g_m \cdots g_1 &= s_k \tilde{g}_{n-k} \cdots \tilde{g}_1 f_1^{-1} \cdots f_k^{-1} \\ &= (s_k \tilde{g}_{n-k}) \tilde{g}_{n-k-1} \cdots \tilde{g}_1 \tilde{g}_n \cdots \\ &\quad \circ \tilde{g}_{n-k+2} \big(\tilde{g}_{n-k+1} s_k^{-1}\big), \end{aligned}$$

and applying Theorem 2 we have the result. The case $n < k$ follows analogously.

To prove the second statement of this theorem it is enough to recall that given $s \in \mathcal{S}$, tst stays in $\mathcal{S}$ if and only if s is diagonal. □

It can be noted from the previous theorem that the length of a cyclically-reduced word is an invariant of the conjugacy class. Since a nontrivial, cyclically-reduced word has the same number of elementary and affine terms, we refer to this number as the *semilength* of the word. Theorem 3 also implies that two cyclically-reduced maps that are conjugate have the same polydegree up to cyclic permutations. We will call this sequence the polydegree of the conjugacy class.

3. Involutory reversing symmetries

As was noted in the introduction, a map has an involutory reversing symmetry if and only if it can be expressed as the composition of two involutions. In this section we make use of this property to describe the class of polynomial automorphisms that possess involutory reversors. We start by studying involutions.

3.1. Polynomial involutions

To begin, we show that all polynomial involutions are dynamically trivial, i.e., are conjugate to affine or elementary maps.

Proposition 4. *A map $g \in \mathcal{G}$ is an involution if and only if g is conjugate to an affine or to an elementary involution.*

Proof. Assume that g is an involution conjugate in $\mathcal{G}$ to a cyclically-reduced map with semilength m: $\tilde{g} = a_m e_m \cdots a_1 e_1$, $m \geqslant 1$. As the involution condition is preserved under conjugacy we have,

$$\tilde{g}^2 = a_m e_m \cdots a_1 e_1 a_m e_m \cdots a_1 e_1 = \mathrm{id}.$$

But this is a contradiction since the identity cannot be written as a reduced word. It follows that g must be conjugate to either an affine or to an elementary map. □

We investigate next the affine and elementary involutions. For later use, we also find their normal forms corresponding to conjugacy by elements in $C_{\mathcal{S}}(t)$.

Proposition 5. *In addition to the identity, elementary involutions correspond to the following classes and normal forms under $C_{\mathcal{S}}(t)$-conjugacy.*

(1) $(x, y) \to (-x + p(y), y)$, *$p(y)$ any polynomial.* *Normal form*: $(x, y) \to (-x + p(y), y)$, $p(y) = y^l + O(y^{l-2})$.
(2) $(x, y) \to (x + p(y), -y + \eta)$, *$p(y)$ odd around* $\frac{\eta}{2}$. *Normal form*: $(x, y) \to (x + p(y), -y)$, *$p(y)$ odd with leading coefficient* 1.
(3) $(x, y) \to (-x + p(y), -y + \eta)$, *$p(y)$ even around* $\frac{\eta}{2}$. *Normal form*: $(x, y) \to (-x + p(y), -y)$, *$p(y)$ even with leading coefficient* 1.

These normal forms are unique up to replacing $p(y)$ by $\zeta p(y/\zeta)$, where ζ is any root of unity of order $l-1$ and l is the degree of $p(y)$.

Proof. For an elementary automorphism (5) we have

$$e^2(x, y) = \big(\alpha^2 x + \alpha p(y) + p(\beta y + \eta), \\ \beta^2 y + \beta\eta + \eta\big).$$

Setting $e^2 = \mathrm{id}$ it is easy to see that if $e \neq \mathrm{id}$, e has to be of one of the three classes in the proposition. Now, defining coordinates $u = ax + b$, $v = ay + b$, a simple calculation shows that e can be written in the corresponding normal form. Moreover the values of a and b yielding that normal form are unique up to $(l-1)$th roots of unity. □

Remark. It can be proved that every elementary involution is $\mathcal{E}$-conjugate to one of the affine maps, $(x, y) \to (\pm x, \pm y)$, where the coefficients of x and y are conjugacy invariants. However for the purposes of this Letter we will only normalize the involutions by using conjugacy in $C_{\mathcal{S}}(t)$. Normal forms for elementary maps are fully discussed in [9].

Proposition 6. *Let a be an affine, nonelementary automorphism,*

$$a : (x, y) \to \hat{a}(x, y) + (\xi, \eta), \tag{12}$$

with $\hat{a}$ linear. Then a is an involution if and only if the eigenvalues of $\hat{a}$ are 1 *and* -1 *and (ξ, η) is in the eigenspace of* -1.

Furthermore, all affine nonelementary involutions are $\mathcal{S}$-conjugate to t.

Proof. Let be a given by (12). This map is an involution if for every (x, y),

$$\hat{a}^2(x, y) + (\hat{a} + \mathrm{id})(\xi, \eta) = (x, y).$$

The above identity holds if and only if $\hat{a}$ is an involution and has -1 as eigenvalue, with associated eigenvector (ξ, η). Taking into account that $\hat{a}$ is not elementary, the condition $\hat{a}^2 = \mathrm{id}$ means that the eigenvalues of $\hat{a}$ must be 1 and -1. Besides, it can be noted that the eigenspace of -1 is generated by $(\hat{a} - \mathrm{id})(1, 0)$. This follows from $\hat{a}^2 - \mathrm{id} = (\hat{a} + \mathrm{id})(\hat{a} - \mathrm{id}) = 0$ and the assumption that $a \notin \mathcal{S}$.

To prove the second part of the proposition, consider first a linear, nonelementary involution $\hat{a}(x, y)$. In that case, taking $s(x, y) = x(1,0) + y\hat{a}(1,0)$, we see that $\hat{a} = sts^{-1}$.

Next, we show that every affine, nonelementary involution (12) is $\mathcal{S}$-conjugate to its linear part $\hat{a}$. We know that $(\xi, \eta) = (\hat{a} - \mathrm{id})(c, 0)$ for some scalar c. Taking $s(x, y) = (x + c, y)$ it follows that $sas^{-1} = \hat{a}$ and the proof is complete. □

3.2. *Normal forms*

We intend to describe polynomial automorphisms that are reversible by involutions. Let g be one such automorphism. In that case $g = R_1 R_0$, where R_1 and R_0 are involutions. According to Proposition 4, $R_i = g_i r_i g_i^{-1}$, $i = 0, 1$, where r_i is an elementary or an affine involution and $g_i \in \mathcal{G}$. Then g is conjugate to $r_1 f r_0 f^{-1}$ with $f = g_1^{-1} g_0$. If $f \in \mathcal{S}$, g is conjugate to the composition of a pair of involutions in $\mathcal{A} \cup \mathcal{E}$. Let us consider $f \notin \mathcal{S}$ so that it can be written as a reduced word, $f = f_n \cdots f_1$, $n \geqslant 1$. However,

$$f r_0 f^{-1} = f_n \cdots f_1 r_0 f_1^{-1} \cdots f_n^{-1}, \tag{13}$$

is not reduced if r_0 and f_1 are in the same subgroup $\mathcal{A}$ or $\mathcal{E}$. After reducing (13), we obtain either a map $s_0 \in \mathcal{S}$ conjugate to r_0, or a reduced word

$$f_n \cdots f_k \tilde{r}_0 f_k^{-1} \cdots f_n^{-1},$$

where $\tilde{r}_0$ is an affine or an elementary involution, conjugate to r_0. In the last case we see that g is conjugate to

$$f_k^{-1} \cdots f_n^{-1} r_1 f_n \cdots f_k \tilde{r}_0.$$

Now, $f_k^{-1} \cdots f_n^{-1} r_1 f_n \cdots f_k$ is not necessarily a reduced word. This expression reduces either to a map $s_1 \in \mathcal{S}$ conjugate to r_1, or to a reduced word, $f_k^{-1} \cdots f_{k+l}^{-1} \tilde{r}_1 f_{k+l} \cdots f_k$, where $\tilde{r}_1$ is an affine or an elementary involution, conjugate to r_1. Therefore a polynomial automorphism is reversible if and only if it is conjugate to a cyclically-reduced map of one of the following types:

- (**R1**) (trivial case) $g = \tilde{r}_1 \tilde{r}_0 \in \mathcal{A} \cup \mathcal{E}$, where $\tilde{r}_0$, and $\tilde{r}_1$ are both affine or both elementary involutions;
- (**R2**) a reduced word,

$$g = f_1^{-1} \cdots f_m^{-1} \tilde{r}_1 f_m \cdots f_1 \tilde{r}_0, \tag{14}$$

where $\tilde{r}_0, \tilde{r}_1$ are involutions. Note that this includes g of the form $\tilde{r}_1 \tilde{r}_0$ with $\tilde{r}_0, \tilde{r}_1$ not in the same subgroup $\mathcal{A}$ or $\mathcal{E}$.

An immediate consequence of this structure is that the possible polydegrees for conjugacy classes of reversible maps are restricted to be of the form $([l_0], l_1, \ldots, l_m, [l_{m+1}], l_m, \ldots, l_1)$, the terms in brackets being optional.

Friedland and Milnor [9] obtained normal forms for conjugacy classes in $\mathcal{G}$ using generalized Hénon transformations (2). Note that (2) is of the form te with e an elementary map. If the polynomial $p(y)$ in (2), has leading coefficient equal to 1 and center of mass at 0, i.e., if $p(y) = y^l + O(y^{l-2})$, we say that the Hénon transformation is *normal*. When restricted to the real case we say the transformation is normal if the center of mass of $p(y)$ is 0 and the leading coefficient is ± 1. Now, according to [9, Theorem 2.6], every cyclically reduced map that is not elementary or affine is conjugate to a composition of generalized Hénon transformations. Moreover, with the additional requirement that the Hénon transformations be normal, that composition is unique, up to finitely many choices. It can be noted that the number of Hénon transformations in a Hénon normal form equals the semilength of the word.

For the case of reversible maps, Hénon normal forms do not reflect the specific structure of the word. Our next goal is thus to find normal forms better adapted to reversible automorphisms. Maps of type (**R1**) are dynamically trivial, thus we study conjugacy classes for maps of type (**R2**). Our goal is Theorem 1, which discusses normal forms for reversible automorphisms. The following is a preliminary result.

Lemma 7. *Given a cyclically-reduced map of the form* (9)*, there exist diagonal affine automorphisms $s_m, s_0 \in C_{\mathcal{S}}(t)$, such that $s_m g s_0^{-1}$ is a composition of normal Hénon transformations. In other words,*

$$s_m g s_0^{-1} = t\hat{e}_m \cdots t\hat{e}_1,$$

where each of the terms $t\hat{e}_i$ is a normal Hénon transformation. Additionally the terms in the composition are unique up to a finite number of choices.

Proof. The proof of this result follows closely the methods of [9], so we omit it.

Table 1
Conditions for a Hénon normal form map with semilength $m = 2, 3$, or 4, to be reversible by polynomial involutions

Semilength	Normal form and polydegree	Conditions on δ_i	Conditions on p_i
2	**(AA)** (l_1, l_1)	$\delta_1\delta_2 = 1$	$p_2(y) = c\delta_2 p_1(cy)$, where $c^{l_1+1} = \delta_1$
2	**(EE)** (l_1, l_2)	$\delta_1 = \delta_2 = 1$	None
		$\delta_1 = \delta_2 = -1$	$p_1(y)$ odd
			$p_1(y)$, $p_2(y)$ even
		$\delta_1 = 1, \delta_2 = -1$	$p_1(y)$ even
			$p_1(y)$ odd, $p_2(y)$ even
3	**(EA)** (l_1, l_2, l_1)	$\delta_1\delta_2\delta_3 = 1, \delta_3 = \delta_2^{l_1}$	$p_3(y) = \delta_3 p_1(\frac{y}{\delta_2})$
		$\delta_1\delta_2\delta_3 = 1, \delta_3 = -(-\delta_2)^{l_1}$	$p_2(y)$ odd, $p_3(y) = -\delta_3 p_1(-\frac{y}{\delta_2})$
		$\delta_1\delta_2\delta_3 = -1, \delta_3 = -\delta_2^{l_1}$	$p_2(y)$ even, $p_3(y) = -\delta_3 p_1(\frac{y}{\delta_2})$
4	**(AA)** (l_1, l_2, l_2, l_1)	$\delta_1\delta_2\delta_3\delta_4 = 1$, $c^{l_2+1} = \frac{1}{\delta_3}$ and $c^{l_1+1} = \delta_1^{l_1}\delta_4^{l_1+1}$ for some common c	$p_3(y) = c\delta_3 p_2(cy)$, $p_4(y) = \frac{\delta_4}{c} p_1(\frac{\delta_1\delta_4}{c} y)$
4	**(EE)** (l_1, l_2, l_3, l_2)	$\delta_1\delta_3 = 1, \delta_2\delta_4 = 1, \delta_2 = \delta_1^{l_2}$	$p_4(y) = \delta_4 p_2(\delta_1 y)$
		$\delta_1\delta_3 = 1, \delta_2\delta_4 = 1$, $\delta_2 = -(-\delta_1)^{l_2}$	$p_1(y)$, $p_3(y)$ odd, $p_4(y) = -\delta_4 p_2(-\delta_1 y)$
		$\delta_1\delta_3 = 1, \delta_2\delta_4 = 1$, $\delta_2 = -\delta_1^{l_2}$	$p_1(y)$, $p_3(y)$ even, $p_4(y) = -\delta_4 p_2(\delta_1 y)$
		$\delta_1\delta_3 = 1, \delta_2\delta_4 = -1$, $\delta_2 = \delta_1^{l_2}$	$p_3(y)$ even, $p_4(y) = -\delta_4 p_2(\delta_1 y)$
		$\delta_1\delta_3 = -1, \delta_2\delta_4 = 1$, $\delta_2 = -(-\delta_1)^{l_2}$	$p_1(y)$ odd, $p_3(y)$ even, $p_4(y) = -\delta_4 p_2(-\delta_1 y)$
		$\delta_1\delta_3 = -1, \delta_2\delta_4 = -1$, $\delta_2 = \delta_1^{l_2}$	$p_3(y)$ odd, $p_4(y) = -\delta_4 p_2(\delta_1 y)$

As in their arguments, the normal form is unique up to scaling the polynomials and the parameters δ_i by lth roots of unity, where $l = l_1 \cdots l_{m-1}(l_m - 1)$ and l_i is the degree of e_i. □

With these results, we can recall our main theorem Theorem 1, which is proved in Section 4. The normal forms developed in this theorem give a description of all conjugacy classes for reversible polynomial automorphism. They provide also a way to verify if a given polynomial automorphism is reversible, by checking if it is possible to carry it into any of the normal forms described in the theorem. It would be desirable however to obtain a more direct criteria to distinguish reversible automorphisms. Although that seems difficult in general, we can develop conditions for the shorter words. It is clear, for example, that a normal Hénon transformation $h = te$ is reversible if and only if $\delta = 1$, so that e is a normal involution, or if $\delta = -1$ and $p(y)$ is an even polynomial. These are then the only reversible maps of semilength 1, written as composition of normal Hénon transformations. In Table 1 we summarize the criteria for words of semilength $m = 2, 3$ and 4 when they are given in normal Hénon form, i.e., written as composition of normal Hénon transformations, $g = h_m \cdots h_1$,

$$h_i = te_i : (x, y) \to (y, p_i(y) - \delta_i x),$$
$$i = 1, \ldots, m.$$

The corresponding polydegree is assumed to be $(l_1, \ldots, l_m)$. The conditions in Table 1 should be understood up to cyclic reorderings of the indexes.

4. Proof of Theorem 1

We now proceed to the proof of Theorem 1.

Proof. Consider g given by the reduced word (14). We can replace all affine terms in that expression by $st\tilde{s}$, for some $s, \tilde{s} \in \mathcal{S}$. In particular, if either $\tilde{r}_0$ or $\tilde{r}_1$ are affine we replace them by $s_i t s_i^{-1}$ for some $s_i \in \mathcal{S}$, $i = 0, 1$. This allows us to see that g is conjugate to a word of the form

$$te_1^{-1} \cdots te_m^{-1}[te_{m+1}]te_m \cdots te_1[te_0], \tag{15}$$

where e_0, e_{m+1} are elementary involutions. The brackets around te_0 and te_{m+1} indicate that those terms may not appear, depending on what kind of involutions, affine or elementary, are represented by $\tilde{r}_0$ and $\tilde{r}_1$. Now, this expression can be written as $tf^{-1}t[te_{m+1}]f[te_0]$, with $f = te_m \cdots te_1$. Lemma 7 allows us to replace f by $s_m^{-1}t\hat{e}_m \cdots t\hat{e}_1 s_0$, with each $t\hat{e}_i$ a normal Hénon transformation and $s_m, s_0 \in C_{\mathcal{S}}(t)$. If f^{-1} is similarly replaced, we observe that g is conjugate to a map of the form (15), where the terms te_i are now normal Hénon transformations, for $i = 1, \ldots, m$, and e_0, e_{m+1} are any elementary involutions. Now, if $\tilde{r}_0$ and $\tilde{r}_1$ are affine, so that te_0 and te_{m+1} are omitted in (15), this proves g is conjugate to the normal form (**AA**). However, if either of these involutions are elementary it is possible to make an additional simplification. It should be noted that in such cases the conditions $s_0, s_m \in C_{\mathcal{S}}(t)$ may be unnecessary.

Let us consider the case (**EA**), that is when only one of $\tilde{r}_0, \tilde{r}_1$ is an elementary involution. We can assume that only te_{m+1} appears in (15), the other case being equivalent after a cyclic reordering. We can also assume that the terms te_i, $i = 1, \ldots, m$ are already normal Hénon transformations, but e_{m+1} is an arbitrary involution of any of the classes described in Proposition 5. For $i = 0, \ldots, m$, we introduce diagonal affine automorphisms

$$s_i(x_i, x_{i+1}) = (u_i, u_{i+1}), \quad u_i = a_i x_i + b_i, \tag{16}$$

and replace each term te_i, $i = 1, \ldots, m$ by $t\hat{e}_i = s_i te_i s_{i-1}^{-1}$. Now, if on the one hand $s_0 \in C_{\mathcal{S}}(t)$, and on the other te_i^{-1} is replaced by $\phi(s_{i-1})te_i^{-1}\phi(s_i^{-1})$ for $i = 1, \ldots, m$, with ϕ as given by (7), while te_{m+1} becomes replaced by $\phi(s_m)te_{m+1}s_m^{-1}$, we preserve the conjugacy class and the structure of the word.

For $t\hat{e}_i$, $i = 1, \ldots, m$, to remain normal, it is necessary that $b_i = 0$ and $a_{i+1} = a_i^{l_i}$, $i = 1, \ldots, m$. We also need $a_0 = a_1$ and $b_0 = b_1$ in order to have $s_0 \in C_{\mathcal{S}}(t)$. Finally the condition that $\hat{e}_{m+1}$ be in normal form yields additional equations,

$$\kappa_{m+1} a_m = a_{m+1}^{l_{m+1}}, \qquad l_{m+1}\kappa_{m+1} b_{m+1} = \lambda_{m+1} a_{m+1},$$

where $p_{m+1}(y) = \kappa_{m+1} y^{l_{m+1}} + \lambda_{m+1} y^{l_{m+1}-1} +$ (lower order terms), is the polynomial associated to e_{m+1}. It is not difficult to see that this system of equations gives a_0 up to lth roots of unity, for $l = l_1 \cdots l_{m-1}(l_m l_{m+1} - 1)$. All other a_i and b_i are then uniquely determined.

The remaining case, when both involutions are elementary, follows in a similar way.

We have proved existence of normal forms as promised. Uniqueness of those forms is a consequence of Theorem 3. However, some details deserve further discussion. Note that to preserve the structure of the word as a reversible automorphism, we chose to apply to te_i^{-1} the images under the isomorphism ϕ of the maps s_i, s_{i-1} that modify te_i. To discuss uniqueness we have to check if it is possible to apply other diagonal automorphisms to te_i^{-1}, and still preserve the structure of the word as well as the normalizing conditions.

Suppose that (15) is in normal form, and for $i = 1, \ldots, m$, we replace te_i by $t\hat{e}_i = s_i te_i s_{i-1}^{-1}$, s_i given by (16). Consider also diagonal affine automorphisms $\tilde{s}_i(\tilde{x}_{i+1}, \tilde{x}_i) = (\tilde{u}_{i+1}, \tilde{u}_i)$, with $\tilde{u}_i = \tilde{a}_i \tilde{x}_i + \tilde{b}_i$, and replace te_i^{-1} by $t\tilde{e}_i = \tilde{s}_{i-1} te_i^{-1} \tilde{s}_i^{-1}$, $i = 1, \ldots, m$. It should be noted that when te_0 appears it has to be replaced by $s_0 te_0 \tilde{s}_0^{-1}$, otherwise $\tilde{s}_0$ must be equal to s_0; similar considerations follow with respect to te_{m+1}. If after these changes the word is still in normal form, with no need of cyclic reordering, it would be necessary that $t\tilde{e}_i = t\hat{e}_i^{-1}$ for $i = 1, \ldots, m$. This condition is equivalent to

$$\sigma_i te_i = te_i \sigma_{i-1}, \tag{17}$$

where $\sigma_i = s_i^{-1}\phi(\tilde{s}_i)$,

$$\sigma_i(x, y) = (A_i x + B_i, A_{i+1} y + B_{i+1}), \quad i = 0, \ldots, m,$$

$$A_i = \frac{\tilde{a}_i}{a_i}, \qquad B_i = \frac{\tilde{b}_i - b_i}{a_i}, \quad i = 0, \ldots, m + 1.$$

Then (17) reduces to

$$A_{i+1} = A_{i-1},$$

$$A_{i+1} p_i(y) + B_{i+1} = p_i(A_i y + B_i) - \delta_i B_{i-1},$$

for $i = 1, \ldots, m$. It is not difficult to see that the above equations, together with the requirement that the terms in the composition be in their normal form, imply $B_i = 0$ for all $i = 0, \ldots, m+1$. It also follows that all $A_i = 1$ (so that $\sigma_i = \mathrm{id}$ and $\tilde{s}_i = \phi(s_i)$) unless the subgroup of roots of unity,

$$K = \{\omega \in \mathbb{C}\colon\ p_i(y) = \omega p_i(\omega y),\ i = 0, \ldots, m+1\} \tag{18}$$

is not trivial. Moreover, for normal forms (**EA**) and (**EE**) we also obtain that all $A_i = 1$ when the order of K is odd while if the order of K is even either $A_i = 1$ or $A_i = -1$ for every i.

If for the moment we disregard reorderings, the above discussion allows us to see that the normal forms we have obtained are unique up to the modifications we describe next. Let us consider again that (15) is in normal form and denote by k the order of the group K and by l the number

$$l = l_0 \cdots l_{m-1}(l_m l_{m+1} - 1),$$

with l_0 and l_{m+1} taken equal to 1 if the corresponding involutions do not appear. Let ζ be any lth root of unity when k is odd, and any $2l$th root of unity when k even. For $i = 1, \ldots, m+2$ define $a_i = \zeta^{l_0 \cdots l_{i-1}}$ and set $a_0 = \zeta^{1-l}$. Then all possible normal forms can be obtained by the following modifications:

(1) For $i = 1, \ldots, m+1$ replace the polynomial $p_i(y)$ related to the elementary map e_i, by $a_{i+1} p_i(y/a_i)$ and the coefficient δ_i by $a_{i+1}\delta_i/a_{i-1}$.
(2) If either of the involutions $e_i(x, y) = (p_i(y) - \delta_i x, \epsilon_i y)$, $i = 0$ or $i = m+1$ appears, ϵ_i must be replaced by $\zeta^l \epsilon_i$. Besides $p_0(y)$ has to be replaced by $a_1 p_0(y/\zeta)$ and the coefficient δ_0 by $\zeta^l \delta_0$.

It may be noted that for normal form (**AA**) ζ could be any klth root of unity, although in that case $a_0 = \zeta^{1-(-1)^m l}$. However, some further analysis shows that, depending on the parity of k, it suffices to consider lth or $2l$th roots of unity.

Finally we discuss reorderings. Suppose that $g = g_{2m} \cdots g_1$ is a nontrivial, cyclically-reduced, reversible map. Then it is possible to factor g as the composition of two involutions,

$$\begin{aligned} g &= \left(g_{2m} \cdots \left(g_{2k} s^{-1}\right)\right)\left((s g_{2k-1}) \cdots g_1\right) \\ &= \left(f_{k+1}^{-1} \cdots f_m^{-1} r_1 f_m \cdots f_{k+1}\right) \\ &\quad \circ \left(f_{k-1} \cdots f_1 r_0 f_1^{-1} \cdots f_{k-1}^{-1}\right), \end{aligned} \tag{19}$$

with $s \in \mathcal{S}$ and r_0, r_1 involutions. It should be noted that to obtain normal forms we considered a reordering of the terms that makes the last factor in the reduced word (19) of length 1. For any conjugacy class giving rise to normal forms (**AA**) or (**EE**), there are two different reorderings of the terms having such structure, therefore yielding two families of normal forms. For (**EA**) maps we also required that the last factor corresponds to the affine involution, therefore in the general case there is only one possible reordering. Nevertheless, it is possible that (19) can be factored in more than one way as composition of involutions. In other words, the map can have more than two centers of symmetry, reflecting the existence of different families of reversing symmetries. Then other families of normal forms may arise, corresponding to different choices of symmetry centers. □

5. Conclusions

Though every polynomial automorphism is conjugate to a composition of generalized Hénon maps, we have argued that reversible automorphisms are more appropriately written in the normal forms given in Theorem 1. There are three classes of normal forms depending upon whether the basic involutory reversors are both elementary (**EE**) both affine (**AA**) or one of each (**EA**).

This is not a complete classification of reversible automorphisms however, because we have assumed that the reversors are involutions. Since the techniques for studying the noninvolutory case are considerably different and add complexity to the arguments that are not needed in the more common involutory case treated here, we plan to treat the more general case in a forthcoming paper [12].

There are a number of interesting questions that we have not investigated. The automorphisms with polydegree $(l_1, \ldots, l_n)$ form a manifold of dimension $\sum l_i + 6$ [9]. What about the subset of reversible automorphisms? We have investigated automorphisms that are reversible in $\mathcal{G}$. What can one say about reversible automorphisms that do not have reversors in $\mathcal{G}$?

We also have not yet investigated general properties of the dynamics of reversible maps. The normal forms of Theorem 1 are useful (just as those of Friedland

and Milnor for the nonreversible case) because the number of parameters is considerably smaller. For example, it is easy to see that the number of symmetric fixed points for a reversible map given in normal form (**AA**), (**EA**) or (**EE**) is bounded by the product of the degrees of the polynomials p_i, $i = 0, \ldots, m+1$, while on the other hand (see [9, Theorem 3.1]), the number of fixed points is bounded by the degree of the map. Moreover, normal forms are useful for the study of bifurcations since this structure holds as well for families of involutory reversible maps. Apart from symmetric orbits and their bifurcations, are there other dynamical properties that distinguish the reversible automorphisms? Are there dynamical differences between the three classes of reversible maps?

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