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Reversors and symmetries for polynomial automorphisms of the complex plane

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Abstract
We obtain normal forms for symmetric and reversible polynomial automorphisms (polynomial maps that have polynomial inverses) of the complex and real planes. Our normal forms are based on the Hénon normal form of Friedland and Milnor. We restrict ourselves to the case where the symmetries and reversors are also polynomial automorphisms. We show that each such reversor has finite order and that for nontrivial, real maps, the reversor has order 2 or 4. The normal forms are shown to be unique up to finitely many choices. We investigate some of the dynamical consequences of reversibility, especially for the case where the reversor is not an involution.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Polynomial maps provide one of the simplest, nontrivial classes of nonlinear dynamical systems. A subset of these, called polynomial automorphisms, have polynomial inverses—thus these maps are diffeomorphisms. Since this subset is closed under composition, polynomial automorphisms form a group; for the case of automorphisms on $\mathbb{C}^2$ that we consider in this paper, we will denote this group by $\mathcal{G}$. These maps can have quite complicated dynamics, as exemplified by the renowned Hénon quadratic map [1, 2], which is in $\mathcal{G}$. The family of generalized Hénon maps [3], $h : \mathbb{C}^2 \to \mathbb{C}^2$, defined by

$$h(x, y) = (y, p(y) - \delta x) \tag{1}$$

for any polynomial p, is also in $\mathcal{G}$ whenever $\delta \neq 0$ since

$$h^{-1}(x, y) = (\delta^{-1}(p(x) - y), x)$$

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is also polynomial. These maps are area preserving when $\delta = \pm 1$, and orientation preserving when $\delta > 0$.

The structure of $\mathcal{G}$ is well understood, thanks to a classic result of Jung [4] for two-dimensional maps. In this paper we will make use of Jung's theorem to investigate polynomial automorphisms that have symmetries or reversing symmetries in $\mathcal{G}$. Our results also use extensively the Hénon normal form for polynomial automorphisms as compositions of generalized Hénon maps obtained by Friedland and Milnor [3].

In general, a diffeomorphism g has a symmetry if it is conjugate to itself; that is, there exists a diffeomorphism S such that

$$g = S^{-1} g S. \tag{2}$$

Similarly, g has a reversing symmetry, or is 'reversible', if it is conjugate to its inverse [5–8]; that is, there exists a diffeomorphism R such that

$$g^{-1} = R^{-1} g R. \tag{3}$$

For example, generalized Hénon maps are reversible when $\delta = 1$, and they have a nontrivial symmetry when there is an $\omega \neq 1$ such that $p(\omega y) = \omega p(y)$. Reversible maps occur often in applications. For example, reversibility often arises for Hamiltonian systems because their phase spaces consist of coordinates q and momenta p, and a transformation that reverses the momenta, $R(q, p) = (q, -p)$, often corresponds to reversal of time. Our goal in this paper is to classify the polynomial automorphisms that satisfy (2) or (3) with $S, R \in \mathcal{G}$.

The basic properties of symmetries and reversors are discussed in [9] and reviewed in [8]. The set of symmetries is a group, denoted as

$$\mathrm{Sym}(g) = \{S \in \mathcal{G} : S^{-1} g S = g\}. \tag{4}$$

In group theory terminology, $\mathrm{Sym}(g)$ is the centralizer of g in $\mathcal{G}$. Note that $\mathrm{Sym}(g)$ always includes the subgroup $\{g^j : j \in \mathbb{Z}\}$. We will say that S is a nontrivial symmetry of g in the case where $S \neq g^j$. If all the symmetries of g are trivial, then $\mathrm{Sym}(g)$ is isomorphic to $\mathbb{Z}$ (or to $\mathbb{Z}_n$ if g has finite order).

Similarly, whenever R is a reversor for g, then so are each of the members of the family

$$\{R_{j,k} = g^j R^{2k+1} : j, k \in \mathbb{Z}\}. \tag{5}$$

Thus, if R is a reversor, then its inverse is one as well. On the other hand, the composition of any two reversors is a symmetry of g (and is not a reversor unless g is an involution). Thus, for example, if R is a reversor, then R^2 is a symmetry. The set of all symmetries and reversing symmetries of g is a group, usually referred as the reversing symmetry group of g

$$\mathrm{Rev}(g) = \{f \in \mathcal{G} : f^{-1} g f = g^{\pm 1}\}. \tag{6}$$

The group $\mathrm{Sym}(g)$ is a normal subgroup of $\mathrm{Rev}(g)$, and if there are nontrivial reversors, then $\mathrm{Rev}(g)/\mathrm{Sym}(g)$ is isomorphic to $\mathbb{Z}_2$, the cyclic group with two elements. The properties of the group of reversing symmetries have been investigated in several recent papers [10–13] as well as the papers in the collection [14].

Reversors that arise in classical physics examples often are involutions, $R^2 = id$, so that R generates a group $\langle R \rangle = \{id, R\} \simeq \mathbb{Z}_2$. If g possesses an involutory reversing symmetry R, then $\mathrm{Rev}(g) = \mathrm{Sym}(g) \rtimes \langle R \rangle$ [10][4].

However, reversors need not be involutions; examples of maps with noninvolutory reversors (called weakly reversible by Sevryuk [6]) were given by Lamb [9]. As we will

[4] The semidirect product of two groups $\mathcal{G} = \mathcal{N} \rtimes_\phi \mathcal{M}$, with homomorphism $\phi : \mathcal{M} \to \mathrm{Aut}(\mathcal{N})$, is defined as the set of pairs (n, m) with product $(n_1, m_1)(n_2, m_2) = (n_1 \phi_{m_1}(n_2), m_1 m_2)$. The product depends on $\mathcal{N}$, $\mathcal{M}$ as well as ϕ. For the cases we consider, $\mathcal{G} = \mathcal{M}\mathcal{N}$ with $\mathcal{M} \cap \mathcal{N} = \{id\}$ and $\mathcal{N}$ normal. Then a homomorphism is given by $\phi_m(n) = mnm^{-1}$.

see in section 4, if $g \in \mathcal{G}$, then any reversor in $\mathrm{Rev}(g)$ has finite order, $R^k = id$. Note that if k were odd, then g would itself be an involution and thus is dynamically trivial; so if we restrict ourselves to nontrivial maps in $\mathcal{G}$, then their reversors have even order. Moreover, for the special case of maps and reversors on $\mathbb{R}^2$, we will show that the order must be 2 or 4. After we submitted this paper, we learned that Roberts and Baake have also announced this particular result for the special case of real maps in $\mathcal{G}$ [13], though we have not seen their proof.

We will also demonstrate that if g has a nontrivial symmetry, then there is a subgroup of $\mathrm{Sym}(g)$ generated by a finite-order map and a 'root' of g. In particular, we will show

Theorem 1. *Suppose g is a polynomial automorphism of $\mathbb{C}^2$ that possesses nontrivial symmetries. Then g is conjugate to a map of the form*

$$G = s^j (H)^q,$$

where H is a composition of generalized Hénon maps (1), s is a diagonal linear map with finite order, $s^k = id$, j is an integer, and either $s \neq id$ or $q \neq 1$. The normal form has commuting symmetries s and H, and $\mathrm{Sym}(G) \supset \langle H \rangle \times \langle s \rangle \simeq \mathbb{Z} \times \mathbb{Z}_k$.

This theorem is stated in a more detailed way in section 3 as theorem 7 and corollary 9. Note that for the real case, s has order at most 2 (this result was also announced in [13]).

In a previous paper we obtained normal forms for the automorphisms that are reversible by an involution in $\mathcal{G}$ [15]. Just as in [3], these normal forms are constructed from compositions of generalized Hénon maps; however, in this case the reversors are introduced by including two involutions in this composition. We showed there that the involutions can be normalized to be either 'elementary' involutions or the simple affine permutation

$$t(x, y) = (y, x). \tag{7}$$

The second major result of this paper is that in the general case, reversible automorphisms also have at least two basic reversors that are also either elementary or affine. In particular we will prove

Theorem 2. *Suppose g is a nontrivial reversible automorphism of $\mathbb{C}^2$. Then g possesses a reversor of order $2n$ in $\mathcal{G}$ and is conjugate to one of the following classes:*

$\mathbf{R_{AA}}$ $\quad \tau_\omega^{-1} H^{-1} \tau_\omega H,$

$\mathbf{R_{AE}}$ $\quad \tau_\omega^{-1} H^{-1} e_2 H,$

$\mathbf{R_{EE}}$ $\quad e_1^{-1} t H^{-1} e_2 H t,$

where

- *the map H is a composition of generalized Hénon transformations (1),*
- *τ_ω is the affine reversor, $\tau_\omega(x, y) = (\omega y, x)$, such that ω is a primitive nth root of unity, and*
- *the maps e_1, e_2 are elementary reversors,*

$$e_i(x, y) = (p_i(y) - \delta_i x, \epsilon_i y),$$

where $p_i(\epsilon_i y) = \delta_i p_i(y)$ and ϵ_i^2, δ_i^2 are primitive nth roots of unity.

As we will see in section 4, the reversor is an involution ($n = 1$) unless the polynomials in the Hénon maps and the elementary transformations satisfy a common scaling condition. In this case we will see that ω, ϵ_i^2 and δ_i^2 lie in subsets of the nth roots of unity that we will construct explicitly.

We also show in section 4 that the maps in theorem 2 can be normalized and that once this is done the normal forms are unique up to finitely many choices. These details are contained in the more complete statement, theorem 12.

We finish the paper with a discussion of some examples and their dynamics.

2. Background

We start by giving some definitions and basic results concerning the algebraic structure of the group $\mathcal{G}$, presenting our notation and reviewing the work of Friedland and Milnor [3].

2.1. Jung's theorem

The group $\mathcal{G}$ is the group of polynomial automorphisms of the complex plane, the set of bijective maps

$$g : (x, y) \to (X(x, y), Y(x, y)), \qquad X, Y \in \mathbb{C}[x, y],$$

having a polynomial inverse. Here, $\mathbb{C}[x, y]$ is the ring of polynomials in the variables x and y, with coefficients in $\mathbb{C}$. In general, we consider the complex case, but in some instances we will restrict to the case of real maps. The degree of g is defined as the largest of the degrees of X and Y.

The subgroup $\mathcal{E} \subset \mathcal{G}$ of elementary (or triangular) maps consists of maps of the form

$$e : (x, y) \to (\alpha x + p(y), \beta y + \eta), \tag{8}$$

where $\alpha\beta \neq 0$ and $p(y)$ is any polynomial. The subgroup of affine automorphisms is denoted by $\mathcal{A}$. The affine-elementary maps will be denoted by $\mathcal{S} = \mathcal{A} \cap \mathcal{E}$.

We let $\hat{\mathcal{S}}$ denote the group of diagonal affine automorphisms

$$\hat{s} : (x, y) \to (\alpha x + \xi, \beta y + \eta). \tag{9}$$

The group $\hat{\mathcal{S}}$ is the largest subgroup of $\mathcal{S}$ normalized by the permutation t (7), i.e. such that $t\hat{\mathcal{S}}t = \hat{\mathcal{S}}$. On the other hand, the centralizer of t in $\mathcal{S}$ is the subgroup of maps that commute with t,

$$C_{\mathcal{S}}(t) = \{s \in \mathcal{S} : sts^{-1} = t\}.$$

These are the diagonal automorphisms (9) with $\alpha = \beta$ and $\xi = \eta$. Conjugacy by t will be denoted by ϕ,

$$\phi(g) = tgt. \tag{10}$$

Thus, if $s \in C_{\mathcal{S}}(t)$, then $\phi(s) = s$.

According to Jung's theorem [4], every polynomial automorphism $g \notin \mathcal{S}$ can be written as

$$g = g_m g_{m-1} \cdots g_2 g_1, \quad g_i \in (\mathcal{E} \cup \mathcal{A}) \setminus \mathcal{S}, \qquad i = 1, \ldots m \tag{11}$$

with consecutive terms belonging to different subgroups $\mathcal{A}$ or $\mathcal{E}$. An expression of the form (11) is called a reduced word of length m. An important property of a map written in this form is that its degree is the product of the degrees of the terms in the composition [3, theorem 2.1]. A consequence of this fact is that the identity cannot be expressed as a reduced word [3, corollary 2.1]. This means that $\mathcal{G}$ is the free product of $\mathcal{E}$ and $\mathcal{A}$ amalgamated along $\mathcal{S}$. The structure of $\mathcal{G}$ as an amalgamated free product determines the way in which reduced words that correspond to the same polynomial automorphism are related.

Theorem 3 (cf [3, corollary 2.3], or [16, theorem 4.4]). *Two reduced words $g_m \cdots g_1$ and $\tilde{g}_n \cdots \tilde{g}_1$ represent the same polynomial automorphism g if and only if $n = m$ and there exist maps $s_i \in \mathcal{S}$, $i = 0, \ldots, m$ such that $s_0 = s_m = id$ and $\tilde{g}_i = s_i g_i s_{i-1}^{-1}$.*

From this theorem it follows that the length of a reduced word (11) as well as the degrees of its terms are uniquely determined by g. The sequence of degrees $(l_1, \ldots, l_n)$ corresponding to the maps $(g_1, \ldots, g_m)$, after eliminating the 1s coming from affine terms, is referred to as the polydegree of g.

A map is said to be cyclically reduced in the trivial case where it belongs to $\mathcal{A} \cup \mathcal{E}$ or when it can be written as a reduced word (11) with $m \geqslant 2$ and g_m, g_1 not in the same subgroup $\mathcal{E}$ or $\mathcal{A}$.

2.2. Conjugacies of polynomial autormorphims

Two maps $g, \tilde{g} \in \mathcal{G}$ are conjugate in $\mathcal{G}$ if there exists $f \in \mathcal{G}$ such that $g = f\tilde{g}f^{-1}$. If f belongs to some subgroup $\mathcal{F}$ of $\mathcal{G}$, we say that g and $\tilde{g}$ are $\mathcal{F}$-conjugate. It can be seen easily that every $g \in \mathcal{G}$ is conjugate to a cyclically reduced map. Moreover, an explicit calculation shows that every affine map a can be written as $a = st\tilde{s}$, where t is given by (7) and $s, \tilde{s}$ are affine-elementary maps. From these facts it follows that every polynomial automorphism is either trivial (i.e. conjugate to an elementary or an affine map) or is conjugate to a reduced word of the form

$$g = te_m \cdots te_2te_1, \qquad e_i \in \mathcal{E} \setminus \mathcal{S}, \quad i = 1, \ldots, m, \quad m \geqslant 1. \tag{12}$$

Moreover, this representative of the conjugacy class is unique up to modifications of the maps e_i by diagonal affine automorphisms and cyclic reordering. More precisely, we have the following theorem (following [16, theorem 4.6]).

Theorem 4. *Two nontrivial, cyclically reduced words $g = g_m \cdots g_1$ and $\tilde{g} = \tilde{g}_n \cdots \tilde{g}_1$ are $\mathcal{G}$-conjugate if and only if $m = n$ and there exist automorphisms $s_i \in \mathcal{S}, i = 0, \ldots, m$ with $s_m \equiv s_0$ and a cyclic permutation*

$$(\hat{g}_m, \ldots, \hat{g}_1) = (\tilde{g}_k, \ldots, \tilde{g}_1, \tilde{g}_m, \ldots, \tilde{g}_{k+1})$$

such that $\hat{g}_i = s_i g_i s_{i-1}^{-1}$. In that case,

$$s_0 g s_0^{-1} = \hat{g}_m \cdots \hat{g}_1.$$

In particular, if $g = te_m \cdots te_1$ and $\tilde{g} = t\tilde{e}_m \cdots t\tilde{e}_1$ are conjugate, there exist diagonal automorphisms $s_i \in \hat{\mathcal{S}}$, $s_m \equiv s_0$, and a cyclic reordering

$$(\hat{e}_m, \ldots, \hat{e}_1) = (\tilde{e}_k, \ldots, \tilde{e}_1, \tilde{e}_m \ldots, \tilde{e}_{k+1})$$

such that $t\hat{e}_i = s_i t e_i s_{i-1}^{-1}$ and

$$s_0 g s_0^{-1} = t\hat{e}_m \cdots t\hat{e}_1.$$

Proof. Let $g = g_m \cdots g_1$ and $\tilde{g} = \tilde{g}_n \cdots \tilde{g}_1$ be two nontrivial, cyclically reduced conjugate words. By assumption, there is a reduced word $f = f_k \cdots f_1 \in \mathcal{G}$ such that $g = f\tilde{g}f^{-1}$. Then,

$$g_m \cdots g_1 = f_k \cdots f_1 \tilde{g}_n \cdots \tilde{g}_1 f_1^{-1} \cdots f_k^{-1}. \tag{13}$$

However, the word on the right-hand side of (13) is not reduced. Since $\tilde{g}$ is cyclically reduced, we can suppose, with no loss of generality, that f_1 and $\tilde{g}_1$ belong to the same subgroup $\mathcal{A}$ or $\mathcal{E}$, so that f_1 and $\tilde{g}_n$ lie in different subgroups. Taking into account theorem 3 and the fact that (13) represents a cyclically reduced map, we can reduce (13) to obtain

$$\tilde{g}_n \cdots \tilde{g}_1 f_1^{-1} \cdots f_k^{-1} = \begin{cases} \tilde{g}_n \cdots \tilde{g}_{k+1} \tilde{s}_k & \text{if } n \geqslant k, \\ \tilde{s}_n f_{n+1}^{-1} \cdots f_k^{-1} & \text{if } n < k, \end{cases} \tag{14}$$

where $\tilde{s}_n, \tilde{s}_k \in \mathcal{S}$. Moreover, there exist $\tilde{s}_i \in \mathcal{S}$, $\tilde{s}_0 = id$ such that $\tilde{g}_i \tilde{s}_{i-1} f_i^{-1} = \tilde{s}_i$, for $i = 1, \ldots, \min(n, k)$.

For the case $n \geqslant k$,

$$\begin{aligned} g_m \cdots g_1 &= f_k \cdots f_1 \tilde{g}_n \cdots \tilde{g}_{k+1} \tilde{s}_k \\ &= (\tilde{s}_k^{-1} \tilde{g}_k) \tilde{g}_{k-1} \cdots \tilde{g}_1 \tilde{g}_n \cdots \tilde{g}_{k+2} (\tilde{g}_{k+1} \tilde{s}_k) \end{aligned}$$

and applying theorem 3, we have the result. The case $n < k$ follows analogously.

To prove the second statement of this theorem, it is enough to recall that given $s \in \mathcal{S}$, tst stays in $\mathcal{S}$ if and only if s is diagonal. □

This implies that the length of a cyclically reduced word is an invariant of the conjugacy class. Since a nontrivial, cyclically reduced word has the same number of elementary and affine terms, we refer to this number as the semilength of the word. Theorem 4 also implies that two cyclically reduced maps that are conjugate have the same polydegree up to cyclic permutations. We will call this sequence the polydegree of the conjugacy class.

2.3. Generalized Hénon transformations

A generalized Hénon transformation is any map of the form (1) where $\delta \neq 0$ and $p(y)$ is a polynomial of degree $l \geqslant 2$. Note that a generalized Hénon transformation can be written as the composition

$$h = te, \qquad e(x, y) = (p(y) - \delta x, y).$$

If $p(y)$ has leading coefficient equal to 1 (± 1 in the case of real automorphisms) and centre of mass at 0,

$$p(y) = y^l + \mathrm{O}(y^{l-2}), \tag{15}$$

we say that the polynomial is normal and consequently that the Hénon transformation is normalized. Friedland and Milnor [3] obtained normal forms for conjugacy classes of elements in $\mathcal{G}$ in terms of generalized Hénon transformations; we will call this a Hénon normal form.

Theorem 5 ([3], theorem 2.6). *Every nontrivial $g \in \mathcal{G}$ is conjugate to a Hénon normal form that is a composition of generalized Hénon transformations, $h_m \cdots h_1$. Additionally it can be required that each of the terms h_i be normalized and in that case the resulting normal form is unique, up to finitely many choices.*

To prove this result, it is enough to take theorem 4 into account and check that, given a map $g = te_m \cdots te_1$, it is possible to choose diagonal affine automorphisms s_i, $i = 1, \ldots, m$, in such a way that s_m coincides with s_0, and for every i, $s_i te_i s_{i-1}^{-1}$ is a normalized Hénon transformation. In the next section we will use the following generalization of theorem 5.

Lemma 6. *Given a cyclically reduced map of the form (12), there exist diagonal affine automorphisms $s_m, s_0 \in C_{\mathcal{S}}(t)$, such that*

$$s_m g s_0^{-1} = h_m \cdots h_1,$$

where every term h_i is a normal Hénon transformation. The Hénon maps are unique up to finitely many choices.

Proof. Consider a cyclically reduced map (12), with

$$te_i : (x, y) \to (\beta_i y + \eta_i, \alpha_i x + p_i(y)), \qquad i = 1, \ldots, m.$$

We look first for diagonal affine maps s_i, $i = 0, \ldots, m$, with $s_0, s_m \in C_{\mathcal{S}}(t)$ and such that the maps $t\hat{e}_i = s_i te_i s_{i-1}^{-1}$ are Hénon transformations. If we denote

$$s_i(x, y) = (u, v) = (a_i x + b_i, c_i y + d_i),$$

the problem reduces to the set of equations

$$a_0 = c_0, \qquad b_0 = d_0, \qquad a_m = c_m, \qquad b_m = d_m$$

and

$$\beta_i a_i = c_{i-1}, \quad b_i = d_{i-1} - \eta_i a_i, \qquad i = 1, \ldots, m.$$

This system can be solved easily in terms of $2m$ parameters; a particular solution is obtained by choosing $c_0 = \cdots = c_{m-1} = 1$ and $d_0 = \cdots = d_{m-1} = 0$. We can assume now that the terms te_i in (12) are already Hénon maps but not necessarily in normal form. In that case,

$$te_i : (x_{i-1}, x_i) \to (x_i, x_{i+1}), \qquad x_{i+1} = p_i(x_i) - \delta_i x_{i-1}.$$

Setting $s_i(x_i, x_{i+1}) = (y_i, y_{i+1})$, $y_i = a_i x_i + b_i$, and $t\hat{e}_i = s_i(te_i)s_{i-1}^{-1}$, we have,

$$t\hat{e}_i : (y_{i-1}, y_i) \to (y_i, y_{i+1}), \qquad y_{i+1} = \hat{p}_i(y_i) - \hat{\delta}_i y_{i-1},$$
$$\hat{p}_i(y) = a_{i+1} p_i\left(\frac{y - b_i}{a_i}\right) + \text{const}, \qquad \hat{\delta}_i = \frac{a_{i+1}}{a_{i-1}}\delta_i.$$

In order to have leading coefficients equal to 1, we need

$$\kappa_i a_{i+1} = a_i^{l_i}, \qquad i = 1, \ldots, m,$$

where κ_i is the leading coefficient of $p_i(y)$. On the other hand, we require $a_{m+1} = a_m$, $a_1 = a_0$, since by assumption $s_m, s_0 \in C_{\mathcal{S}}(t)$. It is easy to see that these conditions yield a_1 up to lth roots of unity, where $l = l_1 \cdots l_{m-1}(l_m - 1)$. All other a_i are then uniquely determined. Finally, the coefficients b_i, $i = 1, \ldots, m$, can be chosen so that the next to highest order terms are equal to zero, and we set $b_0 = b_1$ and $b_{m+1} = b_m$ to ensure that s_0, s_m are in the centralizer of t.

The above arguments also show that the terms $t\hat{e}_i$ are unique up to replacing the polynomials $\hat{p}_i(y)$ and parameters $\hat{\delta}_i$ with $\zeta^{l_i \cdots l_0}\hat{p}_i(y/\zeta^{l_{i-1}\cdots l_0})$ and $\zeta^{l_i l_{i-1}}\hat{\delta}_i$, respectively, where ζ is any lth root of unity and $l_0 = 1$. □

2.4. *Roots of unity*

As the proof of lemma 6 shows, the normal forms for polynomial automorphisms are unique only up to a scaling by certain roots of unity. In subsequent sections, we will see that symmetric and reversible automorphisms are associated with several subgroups of the roots of unity. In anticipation of these results, we provide some notation for these subgroups.

Let $\mathcal{U}$ be the group of all roots of unity in $\mathbb{C}$ (the points with rational angles on the unit circle) and $\mathcal{U}_n$ be the group of nth roots of unity:

$$\mathcal{U}_n \equiv \{z \in \mathbb{C} : z^n = 1\}. \tag{16}$$

Given a sequence of normal (15) (nonlinear) polynomials $p_1(y), \ldots, p_m(y)$ and some root of unity ζ, define the set $\mathcal{R}(\zeta) = \mathcal{R}(\zeta; p_1(y), \ldots, p_m(y))$ by

$$\mathcal{R}(\zeta) \equiv \{\omega \in \mathbb{C} : \omega p_{2i+1}(\omega y) = \zeta p_{2i+1}(y),\ \omega p_{2i}(\omega y) = p_{2i}(\zeta y),\ 1 \leqslant 2i, 2i+1 \leqslant m\}. \tag{17}$$

It can be observed that if $\omega \in \mathcal{R}(\zeta)$, then $\omega^{-1} \in \mathcal{R}(\zeta^{-1})$, while if $\omega_1 \in \mathcal{R}(\zeta_1)$ and $\omega_2 \in \mathcal{R}(\zeta_2)$, it follows that $\omega_1\omega_2 \in \mathcal{R}(\zeta_1\zeta_2)$. Thus, the set

$$\mathcal{R}_{\mathcal{E}} \equiv \bigcup_{\zeta \in \mathcal{U}} \mathcal{R}(\zeta) \tag{18}$$

is a subgroup of $\mathcal{U}$. Moreover, unless the sequence of polynomials reduces to one monomial, there are only finitely many $\zeta \in \mathcal{U}$ such that $\mathcal{R}(\zeta)$ is nonempty. In this case $\mathcal{R}_{\mathcal{E}}$ has finite order; hence it coincides with one of the groups $\mathcal{U}_n$. On the other hand, given any $\omega \in \mathcal{R}_{\mathcal{E}}$, the definition (17) implies that there is a unique

$$\zeta(\omega) = \zeta \text{ such that } \omega \in \mathcal{R}(\zeta). \tag{19}$$

Now we define the set

$$\mathcal{R}_{\mathcal{A}} \equiv \{\omega \in \mathcal{U} : p_i(\omega y) = \omega p_i(y),\ i = 1, \ldots, m\}. \tag{20}$$

It can be seen that $\mathcal{R}_{\mathcal{A}}$ is a subgroup of the group of roots of unity $\mathcal{U}$. Moreover, $\omega \in \mathcal{R}_{\mathcal{A}}$ iff $\zeta(\omega) = \omega^2$, i.e. iff $\omega \in \mathcal{R}(\omega^2)$, so that $\mathcal{R}_{\mathcal{A}}$ is a subgroup of $\mathcal{R}_{\mathcal{E}}$. Note that $\mathcal{R}_{\mathcal{A}}$ has finite order for any nonempty sequence of (nonlinear) polynomials.

Finally, we define the subgroup $\mathcal{N}$ of $\mathcal{R}_{\mathcal{E}}$ by

$$\mathcal{N} \equiv \{\omega \in \mathbb{C} : \omega p_i(\omega y) = \zeta p_i(y),\ \text{for some } \zeta \in \mathcal{R}_{\mathcal{A}}\} = \bigcup_{\zeta \in \mathcal{R}_{\mathcal{A}}} \mathcal{R}(\zeta) \tag{21}$$

and the subgroup of $\mathcal{R}_{\mathcal{A}}$ defined by

$$\mathcal{N}' \equiv \{\zeta \in \mathcal{R}_{\mathcal{A}} : \mathcal{R}(\zeta) \neq \emptyset\}. \tag{22}$$

These groups have the ordering

$$\mathcal{N}' \subseteq \mathcal{R}_{\mathcal{A}} \subseteq \mathcal{N} \subseteq \mathcal{R}_{\mathcal{E}} \subseteq \mathcal{U}. \tag{23}$$

Example 2.1. Given the three normal polynomials

$$p_1(y) = y^5, \qquad p_2(y) = y^{13} + ay^5, \qquad p_3(y) = y^{21},$$

if $a \neq 0$, we find that $\mathcal{R}(\zeta)$ is nonempty only for $\zeta \in \mathcal{U}_4$ and that

$$\mathcal{N}' = \mathcal{R}_{\mathcal{A}} = \mathcal{U}_4 \subset \mathcal{N} = \mathcal{R}_{\mathcal{E}} = \mathcal{U}_8.$$

On the other hand, if $a = 0$, then

$$\mathcal{N}' = \mathcal{R}_{\mathcal{A}} = \mathcal{U}_4 \subset \mathcal{N} = \mathcal{U}_8 \subset \mathcal{R}_{\mathcal{E}} = \mathcal{U}_{16}.$$

3. Symmetric automorphisms

In this section we investigate the structure of cyclically reduced words that represent polynomial automorphisms possessing nontrivial symmetries. We will show that if g is a nontrivial polynomial automorphism with nontrivial symmetries, then g is conjugate to a map of the form $s^j H^q$. In this form, s is a finite order, affine-elementary symmetry and H is a cyclically reduced symmetry. This decomposition of g gives rise to a subgroup of $\mathrm{Sym}(g)$ isomorphic to $\mathbb{Z} \times \mathbb{Z}_n$ where n is the order of s. Similar subgroups have been found for the particular case of polynomial mappings of generalized standard form [13].

Using theorem 5, we can, with no loss of generality, assume that g is in Hénon normal form.

Theorem 7. *Suppose that $g = h_m h_{m-1} \cdots h_1$ is a polynomial automorphism in Hénon normal form with a nontrivial symmetry in $\mathcal{G}$. Then there exist diagonal linear transformations $s, \tilde{s}$ such that s has finite order and some integer j such that*

$$g = s^j (\tilde{g})^q, \qquad \text{where } \tilde{g} = \tilde{s} h_r \cdots h_1, \tag{24}$$

where $s, \tilde{g}$ are commuting symmetries of g, $m = qr$, and either $s \neq id$ or $q \neq 1$.

Proof. By assumption, there is a nontrivial symmetry f, and by the condition (2), since g is cyclically reduced, f must be either an affine-elementary map or a nontrivial cyclically reduced word. The set $\{g^j f^n : j, n \in \mathbb{Z}\}$ is a subgroup of $\mathrm{Sym}(g)$. By replacing f with some convenient element in that subgroup if necessary and using theorem 4, we may assume that

$f = s_m h_k \cdots h_1$, with s_m a diagonal affine map, and $0 < k \leqslant m$. In this case the relation $fgf^{-1} = g$ becomes

$$s_m h_k \cdots h_1 h_m \cdots h_{k+1} s_m^{-1} = h_m \cdots h_1.$$

Since $h_i = te_i$, theorem 3 then implies the existence of diagonal affine maps s_i, $i = 0 \cdots m-1$, with $s_0 = s_m$ such that

$$h_{i+k} = s_i^{-1} h_i s_{i-1}, \tag{25}$$

where the indices should be understood mod m.

Let $r = \gcd(m, k)$ the greatest common divisor of m and k, and define integers q, p such that $m = qr$, $k = pr$. In this case, there exist integers j and a such that $r = jk - am$, where we may assume $j, a > 0$ [17]. It then follows that $r \equiv jk \bmod m$, so that $h_{r+i} = h_{jk+i}$. Iterating (25), we obtain

$$h_{i+r} = s_{i+(j-1)k}^{-1} \cdots s_{i+k}^{-1} s_i^{-1} (h_i) s_{i-1} s_{i-1+k} \cdots s_{i-1+(j-1)k}.$$

Defining $\tilde{g}_n = h_{nr} h_{nr-1} \cdots h_{(n-1)r+1}$, we then obtain

$$\tilde{g}_{n+1} = \tilde{s}_n^{-1} \tilde{g}_n \tilde{s}_{n-1}, \qquad \text{for } n = 1, \ldots, q-1, \tag{26}$$

where $\tilde{s}_n = s_{nr} s_{nr+k} \cdots s_{nr+(j-1)k}$. Since $m = qr$, we can use induction on (26) to obtain

$$g = \tilde{s}_{q-1}^{-1} \cdots \tilde{s}_1^{-1} \tilde{s}_0^{-1} \tilde{g}^q, \tag{27}$$

where

$$\tilde{g} = \tilde{s}_0 h_r \cdots h_1 = \tilde{s}_0 \tilde{g}_1. \tag{28}$$

The leading affine-elementary map in (27) is actually the jth power of a simpler map since $r \equiv jk \bmod m$ implies that $\tilde{s}_n = s_{njk} s_{(nj+1)k} \cdots s_{((n+1)j-1)k}$. Using this we regroup the q groups of j terms as j groups of q to obtain

$$\begin{aligned}
\tilde{s}_0 \tilde{s}_1 \cdots \tilde{s}_{q-1} &= (s_0 s_k \cdots s_{(j-1)k})(s_{jk} \cdots s_{(2j-1)k}) \cdots (s_{(q-1)jk} \cdots s_{(jq-1)k}) \\
&= (s_0 s_k \cdots s_{(q-1)k})(s_{qk} \cdots s_{(2q-1)k}) \cdots (s_{(j-1)qk} \cdots s_{(jq-1)k}) \\
&= (s_0 s_k \cdots s_{(q-1)k})^j,
\end{aligned}$$

since $qk \equiv 0 \bmod m$. Thus, we have shown that whenever g has a symmetry it has the form (24), with

$$s \equiv s_{(q-1)k}^{-1} \cdots s_k^{-1} s_0^{-1}.$$

We still have to prove that $s \in \mathrm{Sym}(g)$. Using (25) and the fact that $h_{m+i} = h_i$ allows us to obtain

$$h_i = s_{m+i-k}^{-1} \cdots s_{m+i-qk}^{-1} (h_{m+i-qk}) s_{m+i-1-qk} \cdots s_{m+i-1-k}.$$

As $qk \equiv 0 \bmod m$, the above relations imply

$$s_{m+r-qk} \cdots s_{m+r-k} (h_r \cdots h_1) = (h_r \cdots h_1) s_{m-qk} \cdots s_{m-k},$$

which readily yields $s\tilde{g} = \tilde{g}s$. To see that s must be of finite order, note that since g is assumed to be in Hénon normal form, the relation

$$s h_m \cdots h_1 s^{-1} = h_m \cdots h_1$$

means that s is a diagonal linear transformation

$$s : (x, y) \to (a_0 x, a_1 y),$$

which conjugates a normal form to itself. Then, as in lemma 6, the scaling factors a_0, a_1 must be roots of unity, so that s has finite order. Similarly, since each of the s_i are diagonal, so is $\tilde{s}_0$.

To finish, we show that if f is a nontrivial symmetry, then either $s \neq id$ or $q \neq 1$. To see this, we observe that (26) gives

$$\begin{aligned} f &= s_m h_k \cdots h_1 \\ &= s_m(\tilde{s}_{p-1}^{-1} \cdots \tilde{s}_0^{-1})(\tilde{s}_0 h_r \cdots h_1)^p \\ &= s^a \tilde{g}^p, \end{aligned}$$

where we used $k = pr$ and $jk \equiv r \bmod m$. Therefore if $q = 1$, then $m = r = k$ and $p = 1$, so that when $s = id$ then $f = \tilde{g} = g$ is a trivial symmetry. □

Note that since s has finite order and is a diagonal linear map, then if it is real it must be an involution.

From this theorem we see that every polynomial automorphism that possesses nontrivial symmetries and is not the q-fold iteration of a nontrivial automorphism must have nontrivial symmetries conjugate to affine-elementary maps. In the case of a map given in Hénon normal form, it is not difficult to establish the conditions under which the group of such symmetries is nontrivial.

Proposition 8. *If s is an affine-elementary symmetry of the Hénon normal form map $g = h_m \cdots h_1$, then $s = s_\omega$, where*

$$s_\omega(x, y) = \left(\frac{\zeta(\omega)}{\omega}x, \omega y\right), \qquad \omega \in \mathcal{U}_k, \tag{29}$$

where $\mathcal{U}_k = \mathcal{R}_\mathcal{E}$ if m is even and $\mathcal{U}_k = \mathcal{R}_\mathcal{A}$ if m is odd. Thus, the set of affine-elementary symmetries is a cyclic group isomorphic to $\mathcal{U}_k$.

Proof. According to theorem 3, $s \in \mathcal{S}$ is a symmetry if and only if there exist maps $s_i \in \hat{\mathcal{S}}$, $i = 0, \ldots, m$, $s_0 = s_m = s$, such that

$$s_i h_i s_{i-1}^{-1} = h_i. \tag{30}$$

Moreover, given that g is in normal form, each of the maps s_i must be of the form $s_i(x, y) = (a_i x, a_{i+1} y)$. In that case (30) translates into the conditions

$$\begin{aligned} &a_{i-1} = a_{i+1}, \qquad a_0 = a_m, \\ &p_i(a_i x) = a_{i+1} p_i(x). \end{aligned}$$

When m is even, we must have $a_{2k+1} = \omega$ and $a_{2k} = \zeta(\omega)/\omega$ for $\omega \in \mathcal{R}_\mathcal{E}$, and $\zeta(\omega)$ is defined by (19). Thus, $s = s_0$ has the promised form. Since $\zeta(\omega^2) = (\zeta(\omega))^2$, the set of symmetries $s \in \mathcal{S}$ is a cyclic group isomorphic to $\mathcal{R}_\mathcal{E}$.

In a similar way, if m is odd, all a_i must be equal to some $\omega \in \mathcal{R}_\mathcal{A}$, (20), so that $s(x, y) = (\omega x, \omega y)$. Moreover, since $\zeta(\omega) = \omega^2$ for $\omega \in \mathcal{R}_\mathcal{A}$, the symmetries $s \in \mathcal{S}$ are of the promised form, and they form a cyclic group isomorphic to $\mathcal{R}_\mathcal{A}$. □

Finally, as a corollary to these results we can prove a more complete statement of theorem 1.

Corollary 9. *Suppose g is a polynomial automorphism of the plane that possesses nontrivial symmetries. Then g is conjugate to a map of the form*

$$s_\omega^j (H)^q,$$

where $H = h_n h_{n-1} \cdots h_1$ is a composition of normal generalized Hénon maps (1), and s_ω is given by (29), where $\omega \in \mathcal{R}_\mathcal{E}(p_1(y), \ldots, p_n(y))$ if n is even and $\omega \in \mathcal{R}_\mathcal{A}(p_1(y), \ldots, p_n(y))$ if n is odd, and j is an integer. The normal form has commuting symmetries s_ω and H, and either $\omega \neq 1$ or $q \neq 1$.

Proof. According to theorem 7, g is conjugate to $s^j(\tilde{s}H)^q$, where s is a symmetry of H. Using theorem 5, we can conjugate this map with a new diagonal-affine transformation to normalize the map $\tilde{s}H$. The conjugacy commutes with s since both are diagonal, and according to proposition 8, s has the promised form. □

4. Reversible automorphisms

In [15] we described conjugacy classes for polynomial automorphisms that are reversible by involutions. Although the involution condition appears in a natural way in many cases and it was originally one of the ingredients in the definition of reversible systems [5], this requirement can be relaxed. Indeed, many of the features still hold in the more general case (see, e.g. [8] for further discussion). Moreover, all maps with an involutory reversor are reversible in the more general sense. Finally, it is certainly true that there exist reversible maps that do not possess any involution as a reversor; see for example [9]. We begin by showing that the general reversor in $\mathcal{G}$ is conjugate to one that is affine or elementary. Then we will show that every reversor in $\mathcal{G}$ is of finite order. Finally we prove the main theorem.

4.1. Affine and elementary reversors

Lemma 10. *If $g \in \mathcal{G}$ is nontrivial and reversible, then it is conjugate to an automorphism $\tilde{g}$ that has an elementary or an affine reversor. If the semilength of g is odd, then $\tilde{g}$ has an affine reversor.*

Proof. Following (12), we may assume that $g = te_m \cdots te_1$ is written as a cyclically reduced word. Similarly let R be the reduced word for a reversing symmetry of g. Since both g and $g^{-1} = R^{-1}gR$ are cyclically reduced, then unless R is in $\mathcal{A} \cup \mathcal{E}$, it cannot be a cyclically reduced map. Recall that if R is a reversor for g so are the maps Rg^j and that they have the same order as R if this order is even. Thus without loss of generality, we can assume that R is shorter than g; otherwise as in the proof of theorem 4, there is an $\tilde{R} = Rg^j$ for some $j \in \mathbb{Z}$ such that $\tilde{R}$ is shorter than g and is also a reversor. Moreover, by replacing R with Rg if necessary, we can assume that

$$R = s_0^{-1}e_k t \cdots te_1, \quad s_0 \in \hat{\mathcal{S}}, \qquad k < m, \quad s_0 \in \hat{\mathcal{S}}. \tag{31}$$

Following theorem 4 and given that $g = Rg^{-1}R^{-1}$, there must exist maps $s_i \in \hat{\mathcal{S}}$ such that $te_{i^*}^{-1} = s_i te_i s_{i-1}^{-1}$ for $i^* = k - i + 1$ and $i = 1, \ldots, m$, where the indices are taken modulus m. Consider then the map $\hat{g} = fgf^{-1}$, where $f = te_\nu \cdots te_1$, and $k = 2\nu$ or $k = 2\nu + 1$. Then a simple calculation shows that $\hat{g}$ has a reversor $\hat{R} = fRf^{-1}$ that is either $s_\nu^{-1}t$ when k is even or $s_\nu^{-1}e_{\nu+1}$ when k is odd. Thus $\hat{g}$ is conjugate to g and has a reversor in $\mathcal{A} \cup \mathcal{E}$.

Now suppose that the semilength of the conjugacy class is odd and that $g = te_{2k+1} \cdots te_1$ has an elementary reversor, $R = s_0^{-1}e_1$. Reordering terms, we see that the map

$$\hat{g} = te_{k+1} \cdots te_2 te_1 te_{2k+1} \cdots te_{k+2}$$

has an affine reversing symmetry $\hat{R}$, conjugate to some reversor in the family Rg^j. Therefore in the case of conjugacy classes of odd semilength we may always assume affine reversors. □

Remark. Along the same lines of the proof of the previous lemma, it can be seen that if R is any reversor of a cyclically reduced, nontrivial automorphism g, then R can be written as $R_0 g^j$ for some $j \in \mathbb{Z}$ and R_0 shorter than g. Since $(R_0 g^j)^{2\mu+1} g^l = R_0^{2\mu+1} g^{j+l}$, it follows that the reversors generated by $R_0 g^j$ form a subset of the family generated by R_0.

We can go further by conjugating with maps in S and replacing g by its inverse if necessary, so that g can be taken to have the form (12), and R_0 the form (31). Then, a short calculation shows that the reversors generated by R_0 are of the form

$$R = R_0^{2\mu+1} g^j = (s_0^{-1} t s_k^{-2\mu} t s_0^{-1}) e_k t \cdots t e_1 g^j, \qquad s_0, s_k \in \hat{\mathcal{S}},$$

whenever the index $\mu \geqslant 0$. For negative μ, a similar result follows; however, if the reversor has finite order, it is enough to consider only one of these possibilities. Now, since $s_0, s_k \in \hat{\mathcal{S}}$, the term inside the parenthesis in the above expression reduces to an elementary-affine map. When $j \geqslant 0$, no further simplifications are possible, so that the reduced word that represents R has length $2k - 1$ if the length is considered modulus $2m$. For $j < 0$, an additional simplification yields

$$R = (s_0^{-1} t s_k^{-2\mu} t s_0^{-1}) t e_{k+1}^{-1} \cdots e_m^{-1} t g^{j+1},$$

so that, modulus $2m$, the length of the word becomes $2(m - k) + 1$. We can conclude that if any two reversors of a nontrivial, cyclically reduced map belong to a common family, then their modulus $2m$ lengths must coincide or be complementary (i.e. their sum is equal to $2m$).

4.2. Finite-order reversors

We now show that reversors in $\mathcal{G}$ have finite order. Recall that when R is a reversing symmetry for g, each of the maps (5) is also a reversor and that whenever that R has finite order, the order must be even, unless g is an involution.

Theorem 11. *Every polynomial reversor of a nontrivial, polynomial automorphism has finite even order. In the case of real transformations the order is* 2 *or* 4.

Proof. Following theorem 5, we may assume that g is in Hénon normal form and from lemma 10 that R is an affine or an elementary reversing symmetry for g. Now, it is easy to see that if R is affine, $R = s_0^{-1} t$, and if R is elementary, $R = s_0^{-1} e_1$, for some $s_0 \in \hat{\mathcal{S}}$ and $e_1(x, y) = (p_1(y) - \delta_1 x, y)$ a normalized elementary map. The condition $g = Rg^{-1}R^{-1}$ is then equivalent to the existence of diagonal linear maps, $s_i(x, y) = (a_i x, a_{i+1} y)$, $s_m = s_0$, such that

$$t e_{i^*}^{-1} = s_i t e_i s_{i-1}^{-1}, \qquad \text{where} \begin{cases} i^* = m - i + 1, & R \text{ affine}, \\ i^* = m - i + 2, & R \text{ elementary}. \end{cases} \tag{32}$$

Here the indices are understood modulus m. This in turn means that

$$\delta_i \delta_{i^*} = \frac{a_{i-1}}{a_{i+1}} = \frac{a_{(i+1)^*}}{a_{(i-1)^*}}, \quad p_{i^*}(a_i y) = \delta_{i^*} a_{i+1} p_i(y), \qquad i = 1, \ldots, m. \tag{33}$$

Defining $\omega_i = a_i a_{i^*}$, then (33) implies

$$\omega_i = \omega_{i^*}, \quad \omega_{i-1} = \omega_{i+1}, \quad p_i(\omega_i y) = \omega_{i-1} p_i(y), \qquad i = 1, \ldots, m. \tag{34}$$

Thus all the odd ω_i are equal, as are all the even ω_i. Furthermore $\omega_m \equiv \omega_0$. It follows that when R is affine or when the semilength of g is odd, all the ω_i coincide. Then (34) implies that all ω_i are primitive roots of unity of the same order. The proof is complete upon noting that $R^2(x, y) = (x/\omega_0, y/\omega_1)$, which means that R has order $2n$, where n is the order of ω_i. It can be observed that (34) implies that $2n$ must be a factor of $2(l_i l_{i-1} - 1)$, for all indices i. Finally we see that R is real only if $\omega_0, \omega_1 = \pm 1$, so that the order of R is 2 or 4. $\square$

It is not hard to find normal forms for elementary and affine reversors.

Definition (normalized elementary reversor). An elementary map of the form

$$e : (x, y) \to (p(y) - \delta x, \epsilon y), \qquad p(\epsilon y) = \delta p(y) \tag{35}$$

with $p(y)$ a normal polynomial and ϵ^2, δ^2 some primitive nth roots of unity, will be called a normalized elementary reversor of order $2n$. Note that $e^2 = (\delta^2 x, \epsilon^2 y)$.

Definition (normalized affine reversor). Given any primitive nth root of unity ω, the map

$$\tau_\omega : (x, y) \to (\omega y, x) \tag{36}$$

is a normalized affine reversor of order $2n$. Note that $\tau_\omega^2 = (\omega x, \omega y)$.

These normal forms will form the building blocks of the conjugacy classes of reversible automorphisms.

Let us suppose now that the map $g = h_m \cdots h_1$ is in Hénon normal form and that it has either an elementary or an affine reversing symmetry of order $2n$. In that case the proof of theorem 11 implies there exist some primitive nth roots of unity, ω_0 and ω_1, that solve (34). Comparing this with (17), we see that $\omega_1 \in \mathcal{R}(\zeta)$ for $\zeta = \omega_0\omega_1$. Since ω_1 generates $\mathcal{U}_n$, and $\mathcal{R}_\mathcal{E}$ (18) is a group containing $\mathcal{R}(\zeta)$, this implies that $\mathcal{U}_n \subseteq \mathcal{R}_\mathcal{E}$. If in addition the reversor is affine, then $\omega_0 = \omega_1$, so that $\omega_1 \in \mathcal{R}_\mathcal{A}$ (20), which implies that $\mathcal{U}_n \subseteq \mathcal{R}_\mathcal{A}$.

Reversibility imposes stronger conditions than are apparent in (34) for some cases. This occurs for words with even semilength that have an elementary reversor and for words with odd semilength (in this case, as we noted in lemma 10 we can assume that there is an affine reversor). We note that $i^* = i$ for $i = k + 1$ if the reversor is affine and $m = 2k + 1$, while this identity follows for $i = 1, k + 1$ if the reversor is elementary and $m = 2k$. For such indices, (33) implies the existence of some constants $\hat{\epsilon}_i$ and $\hat{\delta}_i$ such that

$$p_i(\hat{\epsilon}_i y) = \hat{\delta}_i p_i(y), \qquad \hat{\epsilon}_i^2 = \omega_i, \quad \hat{\delta}_i^2 = \omega_{i-1}, \tag{37}$$

where the constants ω_i also satisfy (34) for the corresponding indices.

We can use (34) and (37) to construct reversible maps. Let us suppose, for example, that we have k normalized Hénon transformations, $h_1(y), \ldots, h_k(y)$, a normal polynomial $p_{k+1}(y)$, and a nth root of unity $\omega \in \mathcal{R}_\mathcal{A}(p_1(y), \ldots, p_{k+1}(y))$ such that (37) holds for $i = k + 1$ if we set $\omega_k = \omega_{k+1} = \omega$. Then it is possible to choose the coefficient δ_{k+1} and the remaining Hénon transformations in such a way that the map $g = h_{2k+1} \cdots h_1$ has an affine reversing symmetry of order $2n$. Furthermore, the number of possible choices is finite. As similar statements follow in the other cases, we have a way of generating all conjugacy classes for reversible automorphisms. These conditions also enables us to give an explicit description of conjugacy classes for reversible automorphisms as well as to provide normal forms, as we show next.

4.3. Normal form theorem

We are now ready to prove the main result, which was given in the introduction as theorem 2. Given the previous lemmas, we can now restate the result here in more detail.

Theorem 12. *Let g be a nontrivial automorphism that possesses a reversor of order* $2n$*, and* ω *be any primitive nth root of unity. Then g is conjugate to a cyclically reduced map of one of the following classes:*

RAA $\quad \tau_\omega^{-1}(h_1^{-1} \cdots h_k^{-1})\tau_\omega(h_k \cdots h_1), \quad \omega \in \mathcal{R}_\mathcal{A}(p_1(y), \ldots, p_k(y)),$

RAE $\quad \tau_\omega^{-1}(h_1^{-1} \cdots h_k^{-1})e_{k+1}(h_k \cdots h_1), \quad \omega \in \mathcal{R}_\mathcal{A}(p_1(y), \ldots, p_{k+1}(y)),$

REE $\quad e_1^{-1}(t h_2^{-1} \cdots h_k^{-1})e_{k+1}(h_k \cdots h_2 t), \quad \omega \in \mathcal{R}_\mathcal{E}(p_1(y), \ldots, p_{k+1}(y)),$

where

- *the maps h_i are normalized Hénon transformations,*
- *τ_ω is the normalized affine reversor (36), and*
- *if (ω_i) is the sequence defined by $\omega_1 = \omega$, $\omega_2 = \zeta(\omega)/\omega$, $\omega_{i+1} = \omega_{i-1}$, then the maps e_1, e_{k+1} are normal elementary reversors (35), with $\epsilon_i^2 = \omega_i$, $\delta_i^2 = \omega_{i-1}$.*

Furthermore, these normal forms are unique up to finitely many choices.

Conversely, τ_ω is a reversing symmetry for any map having normal form $\mathbf{R_{AA}}$ or $\mathbf{R_{AE}}$, while e_1 is an elementary reversor for any map of the form $\mathbf{R_{EE}}$.

Proof. We consider the conjugacy class of a polynomial automorphism $g = te_m \cdots te_1$, having a reversing symmetry R_0, of order $2n$. We may assume that g is given in Hénon normal form and according to lemma 10 that R_0 is affine or elementary. Moreover, when m is odd it may be assumed that R_0 is affine. Then R_0 is of the form $s_0^{-1}t$ or $s_0^{-1}e_1$ for some scaling $s_0 : (x, y) \to (a_0 x, a_1 y)$. Throughout the present discussion we continue using the notation introduced in theorem 11. In particular we know that the polynomials $p_i(y)$ satisfy (34). Moreover, if R_0 is affine and $m = 2k + 1$, the polynomial $p_{k+1}(y)$ also satisfies (37), while if R_0 is elementary and $m = 2k$, this condition is satisfied by $p_1(y)$ and $p_{k+1}(y)$.

We discuss the case R_0 affine; the case that R_0 is elementary follows in a similar way. Note that if R_0 is affine then $R_0 = \hat{s}_0 \tau_\omega^{-1} \hat{s}_0^{-1}$ for $\omega = a_0 a_1$ and some diagonal linear map $\hat{s}_0$. Letting $m = 2k$ or $m = 2k + 1$, then (32) implies that

$$g = (s_0^{-1}t)(e_1^{-1}t \cdots te_k^{-1}[te_{k+1}^{-1}]s_k te_k \cdots te_1) = R_0 R_1,$$

where the term in brackets is absent if $m = 2k$. Note that R_1 is also a reversing symmetry of order $2n$, conjugate to either $s_k t$ if m is even or to $e_{k+1}^{-1}s_k = \phi(s_{k+1})e_{k+1}$ when m is odd. In the first of these cases we also note that there exists a diagonal linear map $\hat{s}_k$ such that $s_k t = \hat{s}_k \phi(\tau_\omega)\hat{s}_k^{-1}$. In the second case, note that the map $\tilde{e}_{k+1} = e_{k+1}^{-1}s_k$ is an elementary reversor of the form (35), except that the polynomial $p_{k+1}(y)$ may not be normalized. Therefore g is $\mathcal{S}$-conjugate to a map of the form

$$\tau_\omega^{-1}e_1^{-1}t \cdots te_k^{-1}t[e_{k+1}][\tau_\omega]te_k \cdots te_1, \tag{38}$$

where some of the maps te_i have been modified, but only by scalings of their variables, so that the polynomials $p_i(y)$ still have centre of mass at 0. Furthermore, e_{k+1} is a (not necessarily normalized) elementary reversor of the form (35), with $\epsilon_{k+1}^2 = \delta_{k+1}^2 = \omega$. Finally, the brackets indicate the terms that may be omitted, depending on m odd or even.

We can now replace each of the maps te_i, $i = 1, \ldots, k$, with normalized Hénon transformations, as well as the maps $e_i^{-1}t$ with the corresponding inverses, by applying lemma 6 to $f = te_k \cdots te_1$. In this case, the conjugating maps turn out to be linear transformations. Note that the diagonal linear maps that commute with t also commute with any τ_ω and that the only effect of the conjugacies we apply on e_{k+1} is to rescale the polynomial $p_{k+1}(y)$. In this way we obtain a map of the form (38), conjugate to g, where each of the terms te_i, $i = 1, \ldots, k$, is a normalized Hénon transformation. For the even semilength case, this already shows that g is conjugate to a map of the form $\mathbf{R_{AA}}$ for some nth root of unity $\omega \in \mathcal{R}_{\mathcal{A}}(p_1(y), \ldots, p_k(y))$.

When m is odd, we still have to normalize e_{k+1} to make the leading coefficient of $p_{k+1}(y) = 1$. This can be achieved by choosing some convenient scalings

$$s_i : (x, y) \to (a_i x, a_{i+1} y), \qquad i = 0, \ldots, k, \tag{39}$$

$\phi(s_0) = s_0$, to replace te_{k+1} with $\phi(s_k)te_{k+1}s_k^{-1}$ and then each of the terms te_i with $s_i te_i s_{i-1}^{-1}$, while the corresponding te_i^{-1} are replaced by $\phi(s_{i-1})te_i^{-1}\phi(s_i^{-1}) = \phi(s_i te_i s_{i-1}^{-1})^{-1}$. It is not hard to see that appropriate coefficients a_i can be chosen to give the normal form and that they are unique up to lth roots of unity, with $l = l_1 \cdots l_{k-1}(l_k l_{k+1} - 1)$.

We still need to show that we can replace ω in these expressions with any given root of unity of order n and that the forms thus obtained are uniquely determined up to finitely many possibilities.

Let us consider the case of normal form $\mathbf{R_{AA}}$. Using theorem 4, we see that the terms te_i can be modified only by scalings of the variables since we require these terms to stay normal. If we apply to the terms te_i^{-1} the images under the isomorphism ϕ of the transformations we use to modify te_i as we did to obtain the normal forms, the structure of the word is preserved and the parameter ω does not change either. In the more general case we may replace te_i with $t\hat{e}_i = s_i te_i s_{i-1}^{-1}$, s_i given by (39), while for $i = 2, \ldots, k-1$, te_i^{-1} is replaced by $t\hat{e}_i^{-1} = \tilde{s}_{i-1} te_i^{-1} \tilde{s}_i^{-1}$,

$$\tilde{s}_i : (x, y) \to (\tilde{a}_{i+1} x, \tilde{a}_i y), \qquad i = 1, \ldots, k-1.$$

In this case, and if $k \geqslant 2$, it follows that $t(e_k^{-1} t\tau_\omega)$ must be replaced by $t(\hat{e}_k^{-1} t\tau_{\hat{\omega}}) = \tilde{s}_{k-1} te_k^{-1} t\tau_\omega s_k^{-1}$, while $\tau_\omega^{-1} e_1^{-1}$ becomes replaced with $\tau_{\hat{\omega}}^{-1} \hat{e}_1^{-1} = s_0 \tau_\omega^{-1} e_1^{-1} \tilde{s}_1^{-1}$. If $k = 1$ we have to replace $\tau_\omega^{-1} e_1^{-1} t\tau_\omega$ with $\tau_{\hat{\omega}}^{-1} e_1^{-1} t\tau_{\hat{\omega}} = s_0 \tau_\omega^{-1} e_1^{-1} t\tau_\omega s_1^{-1}$.

For the structure of the word to remain unchanged, we need

$$\tilde{a}_1 = a_0, \qquad \lambda_{i-1} = \lambda_{i+1}, \qquad \text{and} \qquad p_i(\lambda_i y) = \lambda_{i+1} p_i(y), \tag{40}$$

where i runs from 1 to k, $\tilde{a}_{k+1}$ is defined to be equal to a_k, and $\lambda_i = \tilde{a}_i / a_i$ for $i = 1, \ldots, k+1$, while λ_0 is just defined to be equal to λ_2. We also note that then $\hat{\omega} = \lambda_1 \lambda_2 \omega$. Now, for the map $\hat{g}$ obtained in this way, to be in normal form, it is also necessary that $\hat{\omega}$ lies in $\mathcal{R}_\mathcal{A} = \mathcal{R}_\mathcal{A}(\hat{p}_1(y), \ldots, \hat{p}_k(y)) = \mathcal{R}_\mathcal{A}(p_1(y), \ldots, p_k(y))$. It follows that if we set $\zeta = \lambda_1 \lambda_2 = \hat{\omega}/\omega$, ζ is also an element of the group $\mathcal{R}_\mathcal{A}$.

We thus have the result that the solutions (λ_1, λ_2) of (40), yielding alternative normal forms for g, are of the form $(\lambda, \zeta/\lambda)$, for some $\zeta \in \mathcal{R}_\mathcal{A}$ and $\lambda \in \mathcal{R}(\zeta)$. Therefore to obtain all possible normal forms it suffices to consider all $\lambda \in \mathcal{N}$ (21) and set $\lambda_1 = \lambda$, $\lambda_2 = \zeta(\lambda)/\lambda$. The requirement that the polynomials $\hat{p}_i(y)$ have leading coefficients equal to 1 allows us to determine the coefficients a_i, $\tilde{a}_i$ up to lth roots of unity for $l = l_1 \cdots l_{k-1}(l_k - 1)$. If $\mathcal{N} = \mathcal{U}_d$ (23), all possible normal forms arise by taking a_1 as any (ld)th root of unity, and λ as a_1^l.

We show now that $\hat{\omega}$ can be chosen as any primitive nth root of unity. Note that the possible $\hat{\omega}$ are of the form $\zeta\omega$ for some $\zeta \in \mathcal{N}'$. We know that for any $\omega \in \mathcal{R}_\mathcal{A}$, $\mathcal{R}(\omega^2)$ is a nonempty set since it contains ω. Let us denote by $\mathcal{R}_\mathcal{A}^2$ the subgroup of $\mathcal{R}_\mathcal{A}$ that consists of elements of the form ω^2, $\omega \in \mathcal{R}_\mathcal{A}$. It then follows that $\mathcal{R}_\mathcal{A}^2$ is a subgroup of $\mathcal{N}'$. Now, given that $\mathcal{R}_\mathcal{A} = \mathcal{U}_r$ for some r, it is not difficult to see that $\mathcal{R}_\mathcal{A}^2$ is a maximal subgroup of $\mathcal{R}_\mathcal{A}$ if r is even, while $\mathcal{R}_\mathcal{A} = \mathcal{R}_\mathcal{A}^2$ if r is odd. In the last case, and in general whenever $\mathcal{N}' = \mathcal{R}_\mathcal{A}$, we see that g can be written in normal form with ω replaced by any rth root of unity and in particular by any primitive nth root of unity.

However, if $\mathcal{R}_\mathcal{A}$ has even order, it is possible that $\mathcal{N}'$ reduces to $\mathcal{R}_\mathcal{A}^2 \neq \mathcal{R}_\mathcal{A}$, and then it is no longer clear that $\hat{\omega}$ can be chosen as an arbitrary nth root of unity. A short calculation shows that $\hat{\omega}$ can still be taken as any nth root of unity as long as the number r/n is even. When r/n is odd, the only admissible nth roots of unity are the numbers $\exp(\mathrm{i}2\pi\nu/n)$, with ν odd. In particular all primitive nth roots of unity are still possible, but $\hat{\omega}$ cannot be taken as equal to 1, i.e. g lacks involutory reversing symmetries associated to this normal form. It is still possible that there exist involutory reversors, though corresponding to a different reordering of the terms, in the case where the map has other families of reversing symmetries.

In the case of normal form $\mathbf{R_{AE}}$, condition (40) should hold for $i = 1, \ldots, k+1$ (although $\tilde{a}_{k+1}$ is not necessarily equal to a_k) if λ_{k+2} is defined to be equal to λ_k. We still have $\hat{\omega} = \lambda_1 \lambda_2 \omega$, but we need in addition that

$$\hat{\epsilon}_{k+1} = \lambda_{k+1}\epsilon_{k+1}, \qquad \hat{\delta}_{k+1} = \lambda_k \delta_{k+1}, \qquad \hat{\epsilon}^2_{k+1} = \hat{\delta}^2_{k+1} = \hat{\omega}.$$

These conditions imply that the solutions (λ_1, λ_2) for (40) yielding normal forms for g must be of the form $\lambda_1 = \lambda_2 = \lambda$ for some $\lambda \in \mathcal{R}_{\mathcal{A}}$. In other words, the set $\mathcal{N}$ becomes $\mathcal{R}_{\mathcal{A}}$, while $\mathcal{N}'$ becomes $\mathcal{R}^2_{\mathcal{A}}$. The statements about admissible $\hat{\omega}$ then follows, basically unchanged.

We can make analogous considerations for normal form $\mathbf{R_{EE}}$. The possible normal forms are obtained in this case by considering any $\lambda \in \mathcal{R}_{\mathcal{E}}$. If we set $\lambda_1 = \lambda$ and $\lambda_2 = \zeta(\lambda)/\lambda$, the remaining λ_i become determined by $\lambda_{i+1} = \lambda_{i-1}$. Once λ is fixed, the requirement that all polynomials $p_i(y)$ be normal determines the coefficients $a_i, \tilde{a}_i$ up to lth roots of unity, $l = l_1 \cdots l_{k-1}(l_k l_{k+1} - 1)$. Additionally, we have

$$\hat{\epsilon}_i = \lambda_i \epsilon_i, \quad \hat{\delta}_i = \lambda_{i-1}\delta_i, \quad \hat{\epsilon}^2_i = \hat{\omega}_i, \quad \hat{\delta}^2_i = \hat{\omega}_{i-1}, \qquad i = 1, k+1.$$

In that case $\hat{\omega}_1 = \lambda_1^2 \omega_1$. Therefore, the possible $\hat{\omega}_1$ are of the form $\zeta\omega_1$, with $\zeta \in \mathcal{R}^2_{\mathcal{E}}$. With respect to which $\hat{\omega}$ are allowed, there follow conclusions similar to those obtained in the case of normal form $\mathbf{R_{AA}}$ after replacing $\mathcal{N}$ with $\mathcal{R}_{\mathcal{E}}$ and $\mathcal{N}'$ with $\mathcal{R}^2_{\mathcal{E}}$.

Finally, the last assertion of the theorem follows by direct calculation. □

Corollary 13. *A map $f \in \mathcal{A} \cup \mathcal{E}$ is a reversing symmetry for some nontrivial, cyclically reduced automorphism $g \in \mathcal{G}$ if and only if f is $\mathcal{S}$-conjugate to either a normalized affine reversor or to a normalized elementary reversor.*

Corollary 14. *Every polynomial involution is conjugate to one of the normal involutions (i) $(x, y) \to (p(y) - x, y)$, $p(y)$ a normal polynomial, (ii) $(x, y) \to (p(y) - x, -y)$, $p(y)$ normal and even, (iii) $(x, y) \to (p(y)+x, -y)$, $p(y)$ normal and odd, and (iv) $(x, y) \to (y, x)$.*

Proof. Every involution is a reversing symmetry for some nontrivial automorphism. □

Corollary 15. *An elementary, nonaffine map*

$$e : (x, y) \to (p(y) - \delta x, \epsilon y + \eta) \tag{41}$$

is a reversing symmetry of a nontrivial, cyclically reduced map in $\mathcal{G}$ if and only if it has finite even order $2n$, $e^2 \in \mathcal{S}$, and ϵ^2, δ^2 are primitive nth roots of unity.

On the other hand, an affine, nonelementary map

$$a : (x, y) \to \hat{a}(x, y) + (\xi, \eta), \tag{42}$$

$\hat{a}$ a linear transformation, is a reversing symmetry of a nontrivial, cyclically reduced map in $\mathcal{G}$ if and only if it has finite even order $2n$ and $a^2 \in \mathcal{S}$.

Proof. That these conditions are necessary follows easily from corollary 13. To see the sufficiency, we prove that elementary (respectively affine) maps satisfying such conditions are $\mathcal{S}$-conjugate to normalized elementary reversors (respectively normalized affine reversors).

Let us consider the case of an elementary map (41) of order $2n$, such that ϵ^2, δ^2 are primitive nth roots of unity and e^2 is an affine transformation. It is not difficult to see that the condition $e^{2n} = id$ implies that if $\eta \neq 0$ then $\epsilon \neq 1$. This observation allows us to prove that e can always be conjugated to an elementary map (41) having $\eta = 0$. Moreover, the conjugating maps can be chosen in $C_{\mathcal{S}}(t)$.

Next, we see that when $\eta = 0$ the conditions $e^2 \in \mathcal{S}$, $e^{2n} = id$ reduce to the fact that $\epsilon^{2n} = \delta^{2n} = 1$ plus the existence of some constants A and B such that

$$
\begin{aligned}
&p(\epsilon y) - \delta p(y) = Ay + B, \\
&A(\delta^{2n-2} + \delta^{2n-4}\epsilon^2 + \cdots + \epsilon^{2n-2}) = 0, \\
&B(\delta^{2n-2} + \delta^{2n-4} + \cdots + 1) = 0.
\end{aligned}
$$

Straightforward calculations then show that it is possible to choose maps in $\mathcal{S}$ that conjugate e to a normalized elementary reversor. It may be interesting to note that if $\epsilon^2 \neq \delta^2$ the condition $e^{2n} = id$ may be omitted and still it can be granted that e is an elementary reversor of order $2n$.

The case of affine reversors can be worked out in a similar way. It is convenient to prove first that, under the given conditions, an affine map is $\mathcal{S}$-conjugate to its linear part. To obtain this result it is useful to treat the cases $n = 1$ and $n \geqslant 2$ separately. If $n \geqslant 2$, the conditions $a^{2n} = id$, $a^2 \in \mathcal{S}$ (a nonelementary) are equivalent to $\hat{a}^2 = \omega(id)$, where $\omega = -\det \hat{a}$ is a root of unity of order n. When $n = 1$ we need, in addition, that the vector (ξ, η) be an eigenvector of $\hat{a}$ with associated eigenvalue -1. Finally, it is not difficult to check that a linear, nonelementary map $\hat{a}$ of order $2n$ that satisfies the hypothesis $\hat{a}^2 \in \mathcal{S}$, is $\mathcal{S}$-conjugate to τ_ω for $\omega = -\det \hat{a}$. □

Corollary 16. *A polynomial automorphism is reversible by involutions in $\mathcal{G}$ if and only if it is conjugate to any of the normal forms $\mathbf{R_{AA}}$, $\mathbf{R_{AE}}$, and $\mathbf{R_{EE}}$, with $\omega = 1$, so that e_1 and e_{k+1} are normal elementary involutions and h_i, $i = 1, \ldots, k$, are arbitrary normal Hénon transformations.*

Corollary 17. *A real polynomial automorphism has real reversors in $\mathcal{G}$ if and only if it is reversible by involutions, so that it is conjugate to one of the (real) normal forms $\mathbf{R_{AA}}$, $\mathbf{R_{AE}}$, and $\mathbf{R_{EE}}$, with $\omega = 1$, or if it has a reversing symmetry of order 4 and is conjugate to a normal form map $\mathbf{R_{AA}}$, with $\omega = -1$, so that the maps h_i represent normal, real, Hénon transformations whose respective polynomials $p_i(y)$ are odd.*

Proof. We noted earlier that the only possible real, reversing symmetries are of order 2 or 4. However, there are no elementary, real, reversing symmetries of order 4 since this would imply that e is of the form (41), with ϵ^2, δ^2 primitive square roots of unity. Therefore the only possible normal form for a real map with a real reversor of order 4 is $\mathbf{R_{AA}}$, with $\omega = -1$. □

5. Examples

In this section we illustrate some of the concepts and results of sections 3 and 4. We present several examples to illustrate the general theory. We do not assert that these examples are necessarily new, they are merely illustrative—examples of maps with nontrivial symmetry groups are well known (see, e.g. [18]). Examples of maps with noninvolutory reversors have been presented before; for example, Lamb found 'modified Townsville' maps with a reversor of order $4n + 2$ for any n—these maps also have involutory reversors [9]. In addition, Roberts and Baake have shown that 'generalized standard maps', a particular case of semilength-2 maps, can have fourth-order reversors [13].

For the case of semilength-1, i.e. a single generalized Hénon map on $\mathbb{C}^2$, (1), the structure of the reversing symmetry group is well known. As shown in proposition 8, there is an affine-elementary symmetry $S(x, y) = (\omega x, \omega y)$ for any $\omega \in \mathcal{R}_\mathcal{A} = \{\omega : p(\omega y) = \omega p(y)\}$. Since we can take ω to generate $\mathcal{R}_\mathcal{A}$, and S and h commute, $\mathrm{Sym}(h) = \langle h\rangle \times \langle S\rangle \simeq \mathbb{Z} \times \mathcal{R}_\mathcal{A}$. For example, when $p(y)$ is odd, the Hénon map has the reflection symmetry $S(x, y) = (-x, -y)$.

Table 1. Conditions for a semilength-2 Hénon normal form map, $g = h_2 h_1$, to have symmetries or be reversible. Here, the symmetry $s_\omega(x, y) = ((\zeta(\omega)/\omega)x, \omega y)$ is order k, and the reversors τ_ω and e_i are order $2k$, where k is the order of ω.

Case	Normal form	Symmetries	Conditions on δ_i	Conditions on $p_i(y)$
			$\omega \in \mathcal{R}_\mathcal{E}(p_1(y), p_2(y))$	
$\mathbf{S_E}$	$s_\omega h_2 h_1$	s_ω	Arbitrary	Arbitrary
$\mathbf{R_{EE}}$	$\hat{e}_1^{-1} t \hat{e}_2 t$	$\hat{e}_1, \hat{e}_2$	$\delta_1^2 = \delta_2^2 = 1$	$p_1(\hat{\epsilon}_1 y) = \delta_1 \hat{\epsilon}_2 p_1(y)$, $\hat{\epsilon}_1^2 = \omega$
				$p_2(\hat{\epsilon}_2 y) = \delta_2 \hat{\epsilon}_1 p_2(y)$
			$\omega \in \mathcal{R}_\mathcal{A}(p_1(y), p_2(y))$	
$\mathbf{S_A}$	$s_\omega h^2$	s_ω, h	$\delta_1 = \delta_2$	$cp_2(cy) = \omega p_1(y)$
$\mathbf{R_{AA}}$	$\tau_\omega^{-1} h^{-1} \tau_\omega h$	τ_ω	$\delta_1 \delta_2 = 1$	$cp_2(cy) = \delta_2 \omega p_1(y)$

The reversible cases of (1) can be obtained using (33). When $\delta \neq \pm 1$, there are no reversors, so that $\mathrm{Rev}(h) = \mathrm{Sym}(h)$. If $\delta = 1$, then h has the involutory reversor t, which commutes with S so that $\mathrm{Rev}(h) = \mathrm{Sym}(h) \rtimes \langle t \rangle = (\langle h \rangle \rtimes \langle t \rangle) \times \langle S \rangle$. The case $\delta = -1$ is reversible with reversor $R(x, y) = (\epsilon y, \epsilon x)$, provided there is a solution of $p(\epsilon y) = -\epsilon p(y)$. There are two possibilities: if $p(y)$ is even, then $\epsilon = -1$ is such a solution, and there is an involutory reversor $R_1(x, y) = -(y, x)$. In this case the group has the same structure as for the case $\delta = 1$, except that t should be replaced by R_1 (for the real case this is the same as table 5 of [13]). Otherwise the reversors, if they exist, are complex and noninvolutory. In this case there is an ϵ such that the group $\langle R \rangle \simeq \mathbb{Z}_{2k}$ gives all affine-elementary symmetries and reversors, where k is the order of $\mathcal{R}_\mathcal{A}$. Thus, we can write $\mathrm{Rev}(h) = \langle h \rangle \rtimes \langle R \rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}_{2k}$.

Thus, we conclude that the symmetries of the generalized Hénon map are:

(a) $\delta = 1 \Rightarrow \mathrm{Rev}(h) = (\langle h \rangle \rtimes \langle t \rangle) \times \langle S \rangle$,
(b) $\delta = -1$ and $p(y)$ is even $\Rightarrow \mathrm{Rev}(h) = (\langle h \rangle \rtimes \langle R_1 \rangle) \times \langle S \rangle$,
(c) $\delta = -1$, k is even, and there is a solution of $p(\epsilon y) = -\epsilon p(y) \Rightarrow \mathrm{Rev}(h) = \langle h \rangle \rtimes \langle R \rangle$,
(d) and in all other cases $\mathrm{Rev}(h) = \mathrm{Sym}(h) = \langle h \rangle \times \langle S \rangle$.

Here $\langle h \rangle \simeq \mathbb{Z}$, $\langle S \rangle \simeq \mathcal{R}_\mathcal{A} \simeq \mathbb{Z}_k$, $\langle t \rangle \simeq \mathbb{Z}_2$, $\langle R_1 \rangle \simeq \mathbb{Z}_2$, and $\langle R \rangle \simeq \mathbb{Z}_{2k}$, where k is the order of $\mathcal{R}_\mathcal{A}$.

Similarly the symmetries for the complex, semilength-2 case, $g = h_2 h_1$, are also easily found; the results are given in table 1. There are two possible forms, $\mathbf{S_E}$ and $\mathbf{S_A}$, corresponding to the groups $\mathcal{R}_\mathcal{E}$ and $\mathcal{R}_\mathcal{A}$, respectively. For example, in case $\mathbf{S_E}$, there is a symmetry $s(x, y) = (\zeta(\omega)x/\omega, \omega y)$ for any $\omega \in \mathcal{R}_\mathcal{E}$. If $\mathcal{R}_\mathcal{A}$ is trivial or $\delta_1 \neq \delta_2$, then there are no other nontrivial symmetries, so that $\mathrm{Sym}(g) \simeq \mathbb{Z} \times \mathcal{R}_\mathcal{E}$. However, when $\delta_1 = \delta_2$, then there can be additional nonaffine symmetries corresponding to case $\mathbf{S_A}$, provided the polynomials $p_1(y)$ and $p_2(y)$ are related by the scaling shown in the table.

According to theorem 12, there are also two possible reversible cases of semilength-2, corresponding to normal forms $\mathbf{R_{AA}}$ and $\mathbf{R_{EE}}$. The conditions for the existence of these can by found by using (33); they are also given in table 1. Thus, for example, when there is a normal form $\mathbf{R_{AA}}$, the polynomials in h_1 and h_2 must be identical up to a scaling. Moreover, if there are noninvolutory reversors, then the group $\mathcal{R}_\mathcal{A}$ must be nontrivial, which implies that $p_i(y) = y q_i(y)$ for some polynomials $q_i(y)$ such that the degrees of their nonzero terms are not coprime.

Whenever a reversible map has a noninvolutory reversor, then it also has nontrivial symmetries. For example, for the case $\mathbf{R_{AA}}$ in table 1, a noninvolutory reversor corresponds to $\omega \in \mathcal{R}_\mathcal{A} \backslash \{1\}$. In this case $\tau_\omega^2 = s_\omega$ is a symmetry since $\omega \in \mathcal{R}_\mathcal{A} \subset \mathcal{R}_\mathcal{E}$. If in addition $\delta_1 = \delta_2 = 1$, then $e^{-1} = e$, and the map has a 'square root' and consequently a symmetry of the form $\tilde{s}h$.

For longer words the degrees of the terms can be useful as an indication of which terms may be centres of symmetry since the normal forms in theorem 12 have polydegrees that are symmetric about the centres and the polydegree of a cyclically reduced word is a conjugacy invariant. However, explicit conditions analogous to those in table 1 are more difficult to write.

Finally, we give several examples to illustrate table 1.

Example 5.1. As an example, consider the composition of two cubic Hénon maps on $\mathbb{C}^2$, $g = h_2h_1$, where

$$h_1(x, y) = (y, y^3 + ay - \delta_1 x), \qquad h_2(x, y) = (y, y^3 - ay - \delta_2 x), \tag{43}$$

for $a \neq 0$. If all the coefficients in the maps are real, then the real plane is an invariant subspace for g, otherwise g is fully complex. For $(p_1(y), p_2(y))$,

$$\mathcal{N}' = \mathcal{R}_{\mathcal{E}}^2 = \{1\} \subset \mathcal{R}_{\mathcal{A}} = \mathcal{R}_{\mathcal{E}} = \mathcal{U}_2.$$

Since $\mathcal{R}_{\mathcal{E}}$ is nontrivial, this system has symmetries of the form $\mathbf{S_E}$; indeed, since $\zeta(\omega) = 1$, the affine transformation $S_1(x, y) = (-x, -y)$ is a symmetry. If $\delta_1 \neq \delta_2$, all symmetries are of the form $S_1^a g^p$, and so $\mathrm{Sym}(g) = \langle g \rangle \times \langle S_1 \rangle \simeq \mathbb{Z} \times \mathbb{Z}_2$. If, however, $\delta_1 = \delta_2 = \delta$, then since $\mathrm{i}p_2(\mathrm{i}y) = p_1(y)$, there are symmetries of the form $\mathbf{S_A}$. We find that $g = (S_2)^2$, with the symmetry $S_2(x, y) = (-\mathrm{i}y, \mathrm{i}p_1(y) - \mathrm{i}\delta x)$. As the group generated by S_2 is still isomorphic to $\mathbb{Z}$, we have $\mathrm{Sym}(g) = \langle S_2 \rangle \times \langle S_1 \rangle \simeq \mathbb{Z} \times \mathbb{Z}_2$.

There are two possible reversible cases for g. When $\delta_i^2 = 1$, table 1 shows that g potentially can be put in normal form $\mathbf{R_{EE}}$. The scaling relations imply that $\delta_1 = \delta_2 = \delta = \pm 1$ for this to be the case. Then $R_1(x, y) = (p_1(y) - \delta x, \delta y)$ is an involutory reversor and generates a family (5) of reversors that contains all reversors with this ordering.

When $\delta_1\delta_2 = 1$, (43) potentially has a reversor with normal form $\mathbf{R_{AA}}$. In this case the scaling relations also require $\delta_1 = \delta_2 = \delta = \pm 1$, and there is a reversor $R_2(x, y) = (-\mathrm{i}\delta y, \mathrm{i}x)$. When $\delta = 1$, R_2 is an involution; however, when $\delta = -1$, it is order 4 and $R_2^2 = S_1$. In both cases, $\delta = \pm 1$, there is an involutory reversor, R_1; therefore, every reversor can be written as the composition of a symmetry and R_1; for example, $R_2 = S_2R_1$.

Thus, we conclude that there are three distinct cases:

(a) $\delta_1 = \delta_2 \in \mathcal{U}_2 \Rightarrow \mathrm{Rev}(g) = (\langle S_2 \rangle \times \langle S_1 \rangle) \rtimes \langle R_1 \rangle$,
(b) $\delta_1 = \delta_2 \notin \mathcal{U}_2 \Rightarrow \mathrm{Rev}(g) = \mathrm{Sym}(g) = \langle S_2 \rangle \times \langle S_1 \rangle$,
(c) $\delta_1 \neq \delta_2 \Rightarrow \mathrm{Rev}(g) = \mathrm{Sym}(g) = \langle g \rangle \times \langle S_1 \rangle$,

where $\langle S_2 \rangle \simeq \mathbb{Z}$, $\langle S_1 \rangle \simeq \mathbb{Z}_2$, and $\langle R_1 \rangle \simeq \mathbb{Z}_2$.

Example 5.2. Let $g = h_2h_1$, where

$$h_1(x, y) = (y, y^3 + x), \qquad h_2(x, y) = (y, y^3 - x).$$

In this case the associated groups of roots of unity are larger:

$$\mathcal{N}' = \mathcal{R}_{\mathcal{A}} = \mathcal{U}_2 \subset \mathcal{R}_{\mathcal{E}}^2 = \mathcal{U}_4 \subset \mathcal{N} = \mathcal{R}_{\mathcal{E}} = \mathcal{U}_8$$

and $\zeta(\omega) = \omega^4$. Table 1 shows that the nontrivial symmetries are generated by $S_1 = (\omega^3 x, \omega y)$ with $\omega = \mathrm{e}^{\mathrm{i}(\pi/4)}$ a primitive, eighth root of unity and $\langle S_1 \rangle \simeq \mathcal{U}_8$. Since $\delta_1 \neq \delta_2$, there are no symmetries of the form $\mathbf{S_A}$. Thus $\mathrm{Sym}(g) = \langle g \rangle \times \langle S_1 \rangle \simeq \mathbb{Z} \times \mathbb{Z}_8$.

Since $\delta_1\delta_2 \neq 1$, g cannot be written in the form $\mathbf{R_{AA}}$; however, it can be written in the form $\mathbf{R_{EE}}$, for $\hat{\epsilon}_1^8 = -1$. The reversors are generated by $R_1(x, y) = (\hat{\epsilon}_1(y^3 + x), \hat{\epsilon}_1^3 y)$, a 16th order reversor. Note that $R_1^2 = S_1$ and that R_1 commutes with S_1. Thus $\mathrm{Rev}(g) = \langle g \rangle \rtimes \langle R_1 \rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}_{16}$.

Note that $\mathcal{R}_{\mathcal{E}}^2 = \mathcal{U}_4$, so that ω in the $\mathbf{R_{AA}}$ normal form may be replaced only by primitive eighth roots of unity. Thus, there are no real normal forms, and though g is reversible in the group of complex automorphisms, it lacks real reversors.

Example 5.3. Consider the case where $g = h^2$, where h is a normal Hénon transformation. Symmetries of the form $\mathbf{S_E}$ correspond to maps $S^j g^p$, with S a generator of the group of affine-elementary symmetries of g, that is $S = s_\omega$ for ω of maximum order in $\mathcal{R}_\mathcal{E}$.

Symmetries of the form $\mathbf{S_A}$ correspond to maps $s\tilde{g}^p$, where s is an affine-elementary symmetry of g and $\tilde{g} = \tilde{s}h$ is a symmetry of g that commutes with s. It turns out that s is also in Sym(h), so that $s = s_\omega$ for some $\omega \in \mathcal{R}_\mathcal{A}$. Moreover, since $h \in$ Sym(g), it follows that $\tilde{s}$ also belongs to Sym(g), so that $\tilde{s} = s_\omega$ for some $\omega \in \mathcal{R}_\mathcal{E}$. Thus, we can conclude that Sym$(g) = \langle S, h\rangle$. If $\mathcal{R}_\mathcal{A} = \mathcal{R}_\mathcal{E}$, then S is a symmetry of h, and so Sym$(g) = \langle S\rangle \times \langle h\rangle \simeq \mathbb{Z}_k \times \mathbb{Z}$. On the other hand, if $\mathcal{R}_\mathcal{A} \neq \mathcal{R}_\mathcal{E}$, S is not a symmetry of h, and unlike the previous examples, Sym(g) is a nonAbelian group. In this case, however, $\langle S\rangle$ is a normal subgroup of Sym(g), so that Sym$(g) = \langle S\rangle \rtimes \langle h\rangle \simeq \mathbb{Z}_k \rtimes \mathbb{Z}$.

According to table 1, the existence of reversors of the form $\mathbf{R_{EE}}$ requires $\delta^2 = 1$ plus some scaling conditions on the polynomial $p(y)$. The associated reversors are in that case of the form

$$R(x, y) = \left(\frac{1}{\hat{\epsilon}_2}(p(y) - \delta x), \frac{1}{\hat{\epsilon}_1} y\right),$$

$\hat{\epsilon}_1, \hat{\epsilon}_2$ as in table 1. For $\delta = 1$, we see that the scaling conditions are trivially satisfied when $\hat{\epsilon}_1 = \hat{\epsilon}_2 = 1$, so that R is an involution. On the other hand, when $\delta = -1$ the scaling conditions are satisfied only when $p(y)$ is odd or even. In that case $\hat{\epsilon}_i \in \{\pm 1\}$, and again R is an involution.

Reversors of the form $\mathbf{R_{AA}}$ exist only if $\delta^2 = 1$ and $p(y)$ satisfies the condition $cp(cy) = \delta\omega p(y)$ with $\omega \in \mathcal{R}_\mathcal{A}$ for some constant c. The associated affine reversing symmetry is then of the form $R(x, y) = (cy/\omega, x/c)$. Again it can be seen that when $\delta = 1$ the scaling relation is trivially satisfied, taking $c = \omega = 1$, while when $\delta = -1$ that relation is satisfied if and only if $p(y)$ is either an odd or an even polynomial.

We thus have the following two possible structures for the group of reversing symmetries of g:

(a) $\delta \neq \pm 1$ or $\delta = -1$ with $p(-y) \neq \pm p(y) \Rightarrow$ Rev$(g) =$ Sym$(g) = \langle S\rangle \rtimes \langle h\rangle$,
(b) $\delta = 1$ or $\delta = -1$ with $p(-y) = \pm p(y) \Rightarrow$ Rev$(g) = (\langle S\rangle \rtimes \langle h\rangle) \rtimes \langle R\rangle$,

where $\langle S\rangle \simeq \mathbb{Z}_k$, $\langle h\rangle \simeq \mathbb{Z}$, $\langle R\rangle \simeq \mathbb{Z}_2$, and k is the order of $\mathcal{R}_\mathcal{E}$. If k is also the order of $\mathcal{R}_\mathcal{A}$, then Sym$(g) = \langle S\rangle \times \langle h\rangle$.

6. Dynamics

The dynamics of a map is affected in a number of ways by the existence of reversing symmetries. In particular, those orbits that are invariant under a reversor share many of the typical properties of the orbits of symplectic maps, i.e. their spectral and bifurcation properties. Although our main interest is to discuss polynomial diffeomorphisms on $\mathbb{C}^2$, we begin with a general discussion and later focus on the polynomial case. We start by briefly reviewing some of the well known implications of reversibility [18, 14].

Let $\mathcal{O}(x)$ denote the orbit of x under a diffeomorphism g. When $R \in$ Rev(g), the symmetry maps orbits into orbits $R(\mathcal{O}(x)) = \mathcal{O}(R(x))$. Thus orbits either come in symmetric pairs or are themselves invariant under R. If R is a reversor, then an orbit and its reflection are generated in reverse order. If $\mathcal{O}(R(x)) = \mathcal{O}(x)$, the orbit is said to be symmetric with respect to R. Observe then that the orbit is symmetric with respect to any of the reversing symmetries in the subgroup $\langle g, R\rangle$ generated by g and R.

If R is a reversor, then for symmetric orbits forward stability implies backward stability. Moreover, there can be no attractors (in the terminology of Conley) that are symmetric under

R; indeed, if A is a symmetric omega-limit set, then it cannot be asymptotically stable [19]. By contrast, an asymmetric orbit can be attracting just as long as its symmetric partner is repelling.

If R is a reversor and x is a point on a symmetric orbit of period n, then the matrix $Dg^n(x)$ is conjugate to its inverse. Thus every multiplier of a periodic symmetric orbit must be accompanied by its reciprocal. When g is real, it follows that eigenvalues other than ± 1 must appear either in pairs (λ, λ^{-1}), with λ real or on the unit circle, or in quadruplets, $(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1})$. Unlike the symplectic case, 1 and -1 may have odd multiplicity. This situation imposes severe restrictions on the motion of eigenvalues for parametrized families of maps: thus if the multiplicity of 1 or -1 is odd, it must continue to remain so, as long as reversibility is preserved. Therefore, whenever 1 and -1 have odd multiplicity, they should persist as eigenvalues. In the plane this means that for families of reversible, orientation-reversing maps, the spectrum is restricted to the set $\{1, -1\}$. In all the other cases families of reversible maps must preserve orientation.

Though generally reversible maps need not be volume preserving, reversible polynomial automorphisms are, since their Jacobians are necessarily constant. In addition, note that maps with the normal form $\mathbf{R_{AA}}$ are orientation preserving. Maps with the normal form $\mathbf{R_{AE}}$ or $\mathbf{R_{EE}}$ can either preserve or reverse orientation.

We denote the fixed set of a map R by

$$\mathrm{Fix}(R) \equiv \{x : R(x) = x\}.$$

If $S \in \mathrm{Sym}(g)$, then its fixed set is an invariant set. In contrast, the fixed sets of reversors are not invariant but contain points on symmetric orbits.

Indeed, as is well known, to look for symmetric periodic orbits it is enough to restrict the search to the set $\mathrm{Fix}(R) \cup \mathrm{Fix}(gR)$ [5, 8, 18, 20]. Therefore if the reversing symmetry has a nontrivial fixed set, it can be used to simplify the computation of periodic points. Indeed, Devaney's original definition of reversibility [5] required that the fixed set of the reversor be a manifold with half the dimension of the phase space. This is the case for maps on the plane that are reversible by orientation-reversing involutions [20]. From this point of view the noninvolutory polynomial reversors we have described are not very interesting since for each of them the associated symmetric orbits reduce to a single fixed point.

We will now show that this is always the case for order 4 reversing symmetries of $\mathbb{R}^2$. In addition, we will recall the result that in this case the symmetric fixed point is hyperbolic [21].

We start by showing that the fixed set of any order 4 transformation of the plane is a point. This is a well known result of Brouwer, who showed that finite period transformations of $\mathbb{R}^2$ are topologically equivalent to either a rotation or to the composite of a rotation and a reflection about a line through the origin [22]. Nevertheless, we present an elementary proof of the local nature of the fixed set similar to that given by MacKay for the case of involutions [20] because this proof provides additional information that we find useful later. To complete the description of the fixed set, some general results on transformation groups due to Smith turn out to be necessary.

Lemma 18. *If R is an order 4 diffeomorphism of $\mathbb{R}^2$, then its fixed set is a point.*

Proof. We start by proving a local result: the fixed points of R must be isolated. Consider first the map $f = R^2$; by assumption, it is an orientation-preserving involution. Suppose that f has a fixed point; in particular, any fixed point of R is a fixed point of f. Without loss of generality, we can choose coordinates so that this point is the origin. We now show that in such a case f must be locally conjugate to $-id$. A simple calculation shows that $Df(0, 0) = \pm I$; thus we can write

$$f : (x, y) \to \pm(x, y) + (f_1(x, y), f_2(x, y))$$

with $f_k(x, y) = o(|(x, y)|)$. Define new variables u and v according to the local diffeomorphism

$$(u, v) = \pm(x, y) + \tfrac{1}{2}(f_1(x, y), f_2(x, y)),$$

where the sign is chosen in accordance with $Df(0, 0)$. In these local variables the map f reduces to $(u, v) \to \pm(u, v)$. Moreover, as this condition holds at each of the fixed points of f, we conclude that if f were locally the identity, then $f = id$ on its domain as long as this domain is connected. Thus if the fixed points of R were not isolated, then $f = id$. However, since R is not an involution by assumption, we see that if R has a fixed point then f is locally conjugate to $-id$. This implies that the Jacobian matrix of R at each fixed point has a (real) normal form given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{44}$$

To prove that R actually has a unique fixed point, we require a classical, global result due to Smith [23]. First extend the map R to $\mathbb{S}^2$ by adding the point at infinity and making it a fixed point. Smith's theory of transformation groups implies that the fixed set of an order 4 map acting on $\mathbb{S}^2$ is either empty or has the homology-mod 2 of $\mathbb{S}^0$, $\mathbb{S}^1$, or $\mathbb{S}^2$. All cases except $\mathbb{S}^0$ can be ruled out, however, because we know that the fixed set of R acting on the sphere is nonempty and that its fixed points (at least other than the fixed point at infinity) are isolated. Therefore the fixed set of R on $\mathbb{S}^2$ must consist exactly of two points. Restricted to the plane, we see that R has exactly one fixed point. □

Note that whenever a reversing symmetry R for some map g has a single fixed point, this point is a symmetric fixed point of g. It was shown by Lamb for the case where R is a rotation by $\pi/2$ that this point cannot be elliptic [21]. Using the previous lemma and its proof, it is easy to generalize this to arbitrary order 4 reversors.

Lemma 19. *If g is a reversible map of $\mathbb{R}^2$ with a real reversor R of order 4, then the associated symmetric fixed point is not elliptic.*

Proof. We may assume that the fixed point is the origin and the Jacobian matrix of R at $(0, 0)$ is given by (44). The reversibility condition implies that

$$DR(0, 0)Dg(0, 0) = Dg(0, 0)^{-1}DR(0, 0).$$

This equation implies that $Dg(0, 0)$ is a symmetric matrix with determinant equal to 1. As symmetric real matrices have real eigenvalues, we conclude that the point $(0, 0)$ cannot be elliptic and that the map is orientation preserving. □

To illustrate some of these phenomena, we give two examples.

Example 6.1. Consider a normal form of type $\mathbf{R_{AA}}$:

$$g = \tau_\omega^{-1} h^{-1} \tau_\omega h,$$
$$h(x, y) = (y, y^3 - by - \delta x), \quad \omega = -1,$$

so that $\tau_\omega(x, y) = (-y, x)$ is an order 4 reversor for g. If we let $p(y) = y^2 - by$ and assume $b \neq 0$, then $\mathcal{R}_A(p) = \mathcal{U}_2$, while $\mathcal{N}' = \{1\}$, and so the only reversing symmetries for this ordering are order 4. Fixed points of this map must satisfy the equations

$$(1 - \delta)x^* = -p(y^*), \qquad (1 - \delta)y^* = p(x^*),$$

where $p(x) = x^3 - bx$. Since the reversor τ_ω has a fixed point at the origin, the origin is always a symmetric fixed point. In general, the stability of a fixed point is determined by

$$\mathrm{Tr}(Dg) = \frac{1}{\delta}(p'(y^*)p'(x^*) + \delta^2 + 1).$$

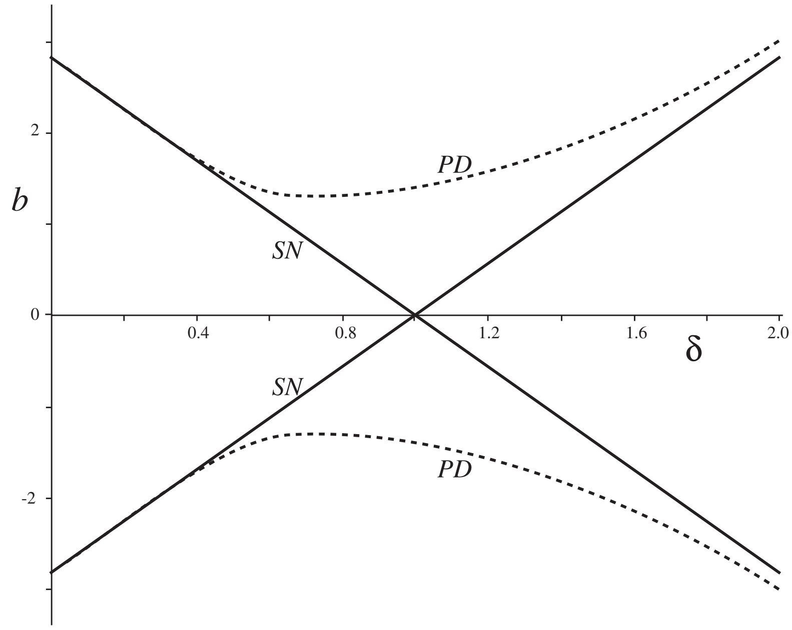

Figure 1. Bifurcation curves for the asymmetric fixed points of (43). The eight asymmetric fixed points exist in the upper and lower quadrants of the cone. They undergo period-doubling bifurcations on the dashed curves.

At the origin this becomes $\mathrm{Tr}(Dg(0,0)) = \delta^{-1}(b^2+\delta^2+1)$, which implies, in accord with lemma 19, that the origin is hyperbolic, since $|\mathrm{Tr}(Dg(0,0)| > 2$, except when $(b,\delta) = (0,\pm 1)$, where it is parabolic.

The remaining eight fixed points are born together in four simultaneous saddle-node bifurcations when

$$b = b_{sn\pm} \equiv \pm 2\sqrt{2}|\delta - 1|.$$

These lines are shown in figure 1; inside the cone $b_{sn-} \leqslant b \leqslant b_{sn+}$, the map g has only one fixed point. The dynamics of this situation are depicted in figure 2 for the case where $\delta = 1.3$.

Outside this cone the map has eight asymmetric fixed points. An example is shown in figure 3; for this case four of the fixed points are elliptic and four are hyperbolic. Note that the four islands surrounding the elliptic fixed points in this figure are mapped into one another by τ_ω. The elliptic fixed points undergo a period-doubling when $\mathrm{Tr}(Dg(x^*, y^*)) = -2$, which corresponds to the curve

$$b^4 - (7 - 13\delta + 7\delta^2)b^2 - 2(2\delta-1)^2(\delta-2)^2 = 0.$$

This gives the dashed curves shown in figure 1. For example, if we fix $\delta = 1.3$ and increase b, then period-doubling occurs at $b \approx 1.6792$. The four new period 2 orbits are stable up to $b \approx 1.7885$, when they too undergo a period-doubling bifurcation. Thus in figure 4, there are four unstable period 2 orbits and four more corresponding period 4 orbits. In this case, the stable and unstable manifolds of the saddles intersect, forming a complex trellis.

In contrast to lemma 19, real maps with order 4 complex reversors can have elliptic symmetric fixed points.

Example 6.2. Consider for instance the polynomial map given in Hénon normal form

$$g = h_2 h_1, \qquad \text{with } h_k(x,y) = (y, p_k(y) + x) \tag{45}$$

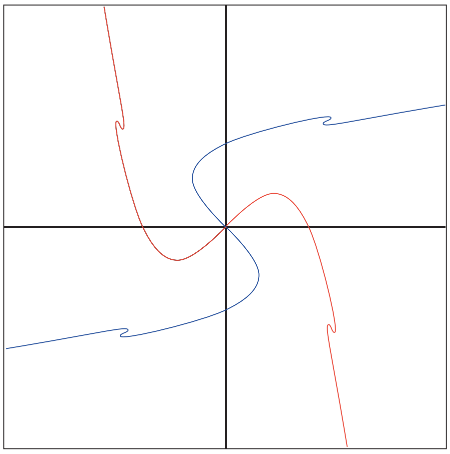

Figure 2. Stable and unstable manifolds of the origin for (43) for $(b, \delta) = (0.85, 1.3)$. The domain of the figure is $(-3, 3) \times (-3, 3)$.

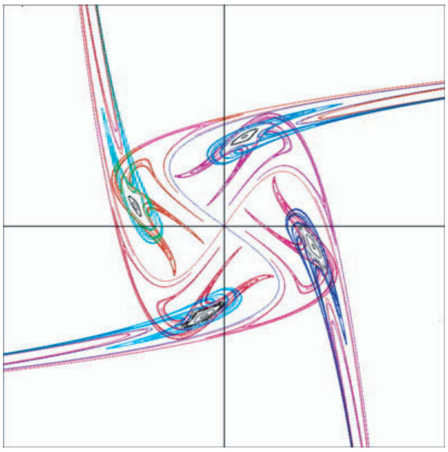

Figure 3. Some orbits of the map (43) for $(b, \delta) = (1.4, 1.3)$. There are elliptic fixed points at $(0.274\,38, 1.211\,6)$ and hyperbolic points at $(1.001\,0, 1.279\,4)$ as well as the images of these points under τ_ω. The domain is the same as figure 2.

and assume that $i \in \mathcal{R}_{\mathcal{A}}(p_1(y), p_2(y))$. According to table 1, g can be written in normal form $\mathbf{R_{EE}}$ with associated order 4 reversors. Direct calculations show that g is conjugate to the map $\hat{g} = t\hat{e}_1^{-1}t\hat{e}_2$, where

$$\hat{e}_1(x, y) = (\hat{p}_1(y) + ix, -iy) \qquad \text{and} \qquad \hat{e}_2(x, y) = (\hat{p}_2(y) - ix, iy)$$

are elementary normal reversors and the $\hat{p}_k$ are rescalings of p_k. Therefore for some scaling s the map $s^{-1}\hat{e}_2 s$ is an order 4 reversor for g and the origin is a symmetric fixed point. Furthermore $\mathrm{Tr}(Dg(0, 0)) = 2 + p_1'(0)p_2'(0)$, so that whenever $-4 < p_1'(0)p_2'(0) < 0$, the origin is an elliptic point. An example is shown in figure 5.

In this example, the map g also possesses involutory reversors. In fact this is always the case for orientation-preserving, semilength 2 maps with normal form $\mathbf{R_{EE}}$. That is, whenever the map has order 4 reversors, there also exist involutory reversors, as can be readily obtained using conditions in table 1.

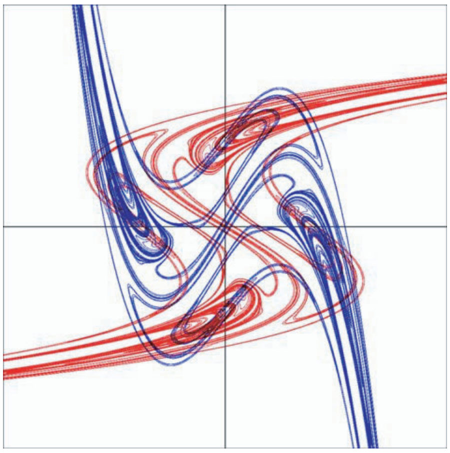

Figure 4. Some stable and unstable manifolds of the map (43) for $(b, \delta) = (1.8, 1.3)$. Here the elliptic points (e.g. at (0.233 90, 1.3607)) have undergone a period-doubling bifurcation. The domain is the same as figure 2.

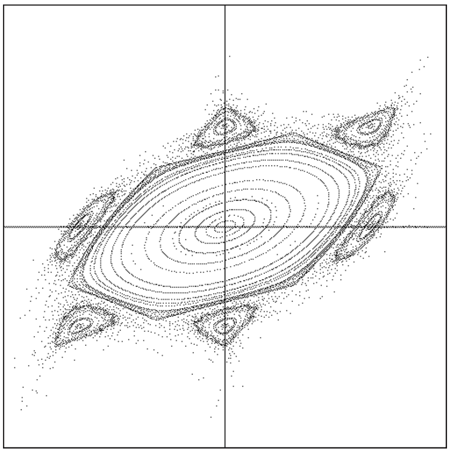

Figure 5. A map of the form (45) with $p_1 = y^5 - 0.5y$ and $p_2 = y^5 + 1.5y$ so that the fixed point at the origin is elliptic. The domain of the figure is $(-1, 1) \times (-1, 1)$.

7. Conclusions

We have shown that maps in $\mathcal{G}$ that have nontrivial symmetries have a normal form $s_\omega^j(H)^q$ in which either there is a finite order, linear symmetry s_ω or in which the map has a root H that is a composite of normal Hénon maps. The symmetry s_ω (29) generates a group isomorphic to $\mathcal{R}_\mathcal{E}$ (18) if the semilength of the map is even and $\mathcal{R}_\mathcal{A}$ (20) if it is odd. This result is encapsulated in corollary 9.

Similarly, we have shown that reversors for automorphisms in $\mathcal{G}$ have normal forms that are either affine or elementary. These can be further normalized, so that the reversors correspond either to the simple affine map τ_ω (36) or to an elementary reversor of the form (35). These reversors have finite, even order. The case that the order is 2, i.e. involutory reversors, is typical in the sense that the existence of reversors of higher order requires that the polynomials in the map satisfy extra conditions so that one of the groups $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{E}}$ is nontrivial. If a map has real reversors, then they must be order 2 or 4.

Using these, we obtained three possible normal forms for reversible polynomial automorphisms of the plane, theorem 12. These correspond to having two affine reversors, two elementary reversors, or one affine and one elementary reversor.

It would be interesting to generalize these results to higher-dimensional polynomial maps. The main difficulty here is that Jung's decomposition theorem has not been generalized to this case. Nevertheless, one could study the class of polynomial maps generated by affine and elementary maps.

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