# Targeting Chaotic Orbits to the Moon Through Recurrence

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**Abstract**

Transport times for a chaotic system are highly sensitive to initial conditions and parameter values. In a previous paper, we presented a technique to find rough orbits (epsilon chains) that achieve a desired transport rapidly. The strategy is to build the epsilon chain from segments of a long orbit – the point is that long orbits have recurrences in neighborhoods where faster orbits must also pass. If a local hyperbolicity condition is satisfied, then a nearby shadow orbit may be constructed with significantly smaller errors. In this paper, we modify the technique to find real orbits, in configuration space, of the restricted three body problem. We find a chaotic Earth – Moon transfer orbit that achieves ballistic capture and that requires 38% less total velocity boost than a comparable Hohmann transfer orbit.

Chaos in a physical system can be exploited to make accessible a wide range of system behaviors without requiring large perturbations. Transport times for a chaotic system are highly sensitive to initial conditions and parameter values. "Targeting" is the process of finding a nearby short path using only available dynamics. In our recent paper [2], we demonstrated an algorithm to reduce transport time for the standard map by a typical factor of $10^4$. It was previously considered particularly difficult to navigate the Hamiltonian dynamics of the standard map [5], due to difficulty in finding short paths between the barriers between resonances in the phase space. In this paper, we apply the algorithm to find short orbits of the restricted three body problem, for parameter values pertaining to the Earth – Moon system.

The algorithm relies on searching for recurrences along the orbit of a dynamical system. We describe the technique for a two-dimensional map:

$$\boldsymbol{z}_{i+1} = T(\boldsymbol{z}_i) \ , \tag{1}$$

which we will derive from the flow of the restricted three body problem by Poincaré section.

Previous targeting algorithms have attempted to find short paths through chaos by looking for them using a straightforward search. The search for example in [4] is analogous to casting out a web of paths leading to the target. Difficulty can arise in searching for paths when the transport from near a starting point $\boldsymbol{a}$ to near the target point $\boldsymbol{b}$ is so slow that the fraction of points which get substantially far from $\boldsymbol{a}$ in a *short* amount of time is so small as to make it virtually invisible to a computer search. In such a case, the web will not leave the region near $\boldsymbol{a}$. This is exactly the situation typified by the layered resonance and barrier structures found in Hamiltonian maps of the plane. [5, 7].

By contrast our algorithm [2] lets a shorter path reveal itself as the "shadow" of an easily found slow orbit which nonetheless makes the desired transport. In brief, suppose that a point on the orbit $\boldsymbol{z}_i$ recurs with $\boldsymbol{z}_{i+s}$, $s$ steps later, i.e. $\|\boldsymbol{z}_i - \boldsymbol{z}_{i+s}\| < \delta$; in this case we can attempt to exploit instability to find a patch that skips the, often very long, recurrent loop. We construct this patch $\boldsymbol{z}'_i$ so that it converges to the preorbit of $\boldsymbol{z}_i$ and to the orbit of $\boldsymbol{z}_{i+s}$. In practice, we choose a moderate value $m$, and require that $\|\boldsymbol{z}'_{i-m} - \boldsymbol{z}_{i-m}\| < \epsilon$ and $\|\boldsymbol{z}'_{i+m} - \boldsymbol{z}_{i+s+m}\| < \epsilon$ for a small preset control constraint $\epsilon$. Thus we attempt to build an $\epsilon$-chain orbit which shadows the $\delta$-chain orbit (consisting of simply skipping the $\delta$ recurrences.)

If the original orbit is hyperbolic, then any point $\boldsymbol{p}$ on an intersection between the unstable manifold of $\boldsymbol{z}_i$ and the stable manifold of $\boldsymbol{z}_{i+s+m}$ has the desired convergence properties, and so may be used as $\boldsymbol{z}'_i$. One technique to obtain such a point is to shoot from the unstable manifold of $\boldsymbol{z}_{i-m}$ well before the recurrence to $\boldsymbol{z}_{i+s+m}$ well after the recurrence. For large enough $m$, and small $\delta$, the curved manifolds are well represented by straight line segments, and the patch orbit will be close to the original orbit.

Thus we require that a point on the unstable direction, $f_u$, at $\boldsymbol{z}_{i-m}$ lands on the stable direction, $f_s$ at $\boldsymbol{z}_{i+s+m}$. That is, we search for an $s$ which solves

$$\left(T^{2m}(\boldsymbol{z}_{i-m} + s f_u) - \boldsymbol{z}_{i+s+m}\right) \times f_s = 0 \ . \tag{2}$$

This can be found by the Newton-secant method.

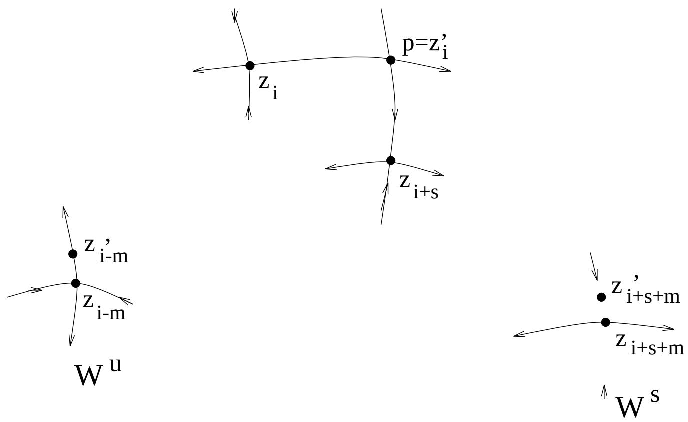


Figure 1: Construction of a patch. When the point $z_i$ recurs with $z_{i+s}$, the point of principal intersection $p$ between $W^u(z_i)$ the unstable manifold of $z_i$ and $W^s(z_{i+s})$ the stable manifold of $z_{i+s}$ converges to the orbit of $z_{i+s}$ under applications of the map $T$, and converges to the preorbit of $z_i$ under applications of the inverse map $T^{-1}$.

To define the stable and unstable directions for an orbit which is not necessarily periodic, we recall that the Jacobian matrix of the map rotates a vector in the tangent space towards the unstable direction, and the Jacobian matrix of the inverse map $T^{-1}$ rotates the vector towards the stable direction. Thus upon iteration, almost any initial unit vector $\boldsymbol{v}$ approaches the unstable direction: $DT^n|_{\boldsymbol{z}_{-n}} \cdot \boldsymbol{v} \equiv DT|_{\boldsymbol{z}_{-1}} \cdot DT|_{\boldsymbol{z}_{-2}} \cdot ... \cdot DT|_{\boldsymbol{z}_{-n}} \cdot \boldsymbol{v} \to f_u(z)$ as $n \to \infty$. Likewise, upon inverse iteration we obtain the stable direction, $DT^{-n}|_{\boldsymbol{z}_n} \cdot \boldsymbol{v} \to f_s(z)$ as $n \to \infty$ and is well approximated for finite $n$, [3, 6]. To accurately calculate these vectors, we renormalize the length to one after each matrix multiplication to prevent the norms from growing (shrinking) beyond machine precision, and we choose a finite value of $n$.

An important modification to our original targeting algorithm arises from the fact that almost all vectors in the tangent space rotate towards the unstable (stable) direction upon repeated forwards (inverse) applications of the tangent maps associated with forward (inverse) orbit. This, of course, is how we find the stable and unstable directions. The implication here is that we do not need to use the true stable and unstable direction vectors in equation Eq. (2). Almost all variations near $\boldsymbol{z}_{i-m}$ will expand along the unstable manifold after $m$ iterates if the patch size $m$ is chosen large enough. A similiar statement can be made regarding variations near

$\boldsymbol{z}_{i+s+m}$ under $m$ inverse iterations. Thus, we will have reduction of the $\delta$ recurrence to within an $\epsilon$ tolerance for almost *any* choice of directions in the place of $f_u$ and $f_s$ in Eq. (2). Specifically, an unstable cone centered around $f_u$ at $\boldsymbol{z}_{i-m}$, depending on $\epsilon$ and $m$, can be defined, within which any vector can be substituted in the place of $f_u$. Similiarly, there exists a stable cone around $f_s$ at $\boldsymbol{z}_{i+s+m}$.

To find a short pseudo-orbit from near $\boldsymbol{a}$ to near $\boldsymbol{b}$ we begin with any orbit, that achieves the transport (within computer memory limits). Then we search for $\delta$ recurrences, attempting to first remove the longest possible recurrent loops, and thus automatically remove intermediate loops for free. The idea is, starting at a point $\boldsymbol{z}_j$, to find the *last* point in the orbit with which it is $\delta$ recurrent. Whenever a $\delta$ recurrence is found, a patch is attempted. If one is found, and it achieves a pre–assigned tolerance $\epsilon$, then the entire (often long) recurrent loop is discarded in favor of the patch, and the next recurrence search begins at the end of the patch. If the recurrence cannot be successfully patched, we continue to search from the end for the next longest recurrence with $\boldsymbol{z}_j$, only incrementing $j$ when no successful patch is found.

In practice, the choice of $\delta$ determines how easily a patch can be found. However the size of $\delta$ also has bearing on the probability of $\delta$ recurrences, and therefore how short the final pseudo-orbit will be. One natural dynamical choice for $\delta$ is related to the size of the turnstiles of the barriers through which the orbit must pass [7]. The true time-optimal orbit will cross each turnstile exactly *once*, in turn. Nonoptimal orbits waste time passing through a given turnstile perhaps many times. However, in practice turnstile sizes are difficult to compute, so we choose $\delta$ large enough so that no opportunities to cut a loop are missed; the only cost of trying to cut a loop for which there exists no patch is wasted computer time.

We will now modify the technique to find real orbits in configuration space (rather than pseudo-orbits in the full phase space) for the planar, circular, restricted three body problem. This problem is the special case of the full three body problem in which one of the masses is taken to be infinitesimal, and so has no influence on the two primaries which are on circular orbits. We normalize the sum of the masses to one, $m_1 = 1 - \mu$ and $m_2 = \mu$, and Newton's gravity constant to one, and use a frame rotating with the primaries, so they are fixed at $x = -\mu$ and $1 - \mu$ respectively. The equations of motion, $\dot{\boldsymbol{w}} = F(\boldsymbol{w})$ for $\boldsymbol{w} = (x, y, u, v)$, are Hill's equation [9].

$$\dot{x} = u ,$$

$$
\begin{aligned}
\dot{y} &= v \,, \\
\dot{u} &= x + 2v - m_1 \frac{x+m_2}{r_1^3} - m_2 \frac{x-m_1}{r_2^3} \,, \\
\dot{v} &= y - 2u - (\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}) y \,,
\end{aligned} \tag{3}
$$

where $r_1^2 = (x+m_2)^2 + y^2$ and $r_2^2 = (x-m_1)^2 + y^2$. The Jacobi integral,

$$
J = u^2 + v^2 - (x^2 + y^2) - 2(\frac{m_1}{r_1} + \frac{m_2}{r_2}) \,, \tag{4}
$$

is a conserved quantity, thus the flow is restricted to a three dimensional submanifold of the four dimensional phase space. On the Poincaré section $y = 0$, $(x, u)$ are equal in value to the canonical variables, and so the map from section to section with $v > 0$ is area preserving.

Our goal here is to look for low energy transfer orbits to the Moon. To this end, we set $m_1/m_2 = 0.0123$. In our coordinates the unit of length is the earth moon distance, $L = 3.844 \cdot 10^5$km, the unit of time is $T = 104h$ and therefore the unit of speed is $V = 1024$m/sec.

The Earth – Moon system has eccentricity 0.055 and so is well approximated by the circular problem. An orbit which becomes a real mission is typically obtained first in such an approximate system and then later refined through more precise models which include effects such as eccentricity, the Sun and other planets, the solar wind, etc. In any case, there is a limited precision to which a rocket can be placed and thrusted so occasional corrective maneuvers are needed. With this in mind, (3) is considered a good starting model [8].

The goal is to beat the energy requirements of the standard Hohmann transfer from a parking orbit around the Earth to a parking orbit around the Moon. This transfer typically takes only a few days, depending on the altitude of the initial parking orbit. It requires two large rocket thrusts (perturbations), one parallel to the motion to leave the Earth, and one antiparallel to the motion to capture the rocket around the Moon. The size of these perturbations, measured by the velocity boost $\Delta V$, depends again on the altitudes of the Earth and Moon orbits. We will see that the chaotic orbit will eliminate the need for the large deceleration at the Moon and reduce required initial boost.

Of course, there is a certain required energy $J_c = -3.1883$, which is that of the Lagrange point $L_2$. This is the minimum energy for which an orbit could possibly move between the primaries. For our mission we set $J = J_0 = -3.17948$ slightly above $J_c$, but below the critical value at which

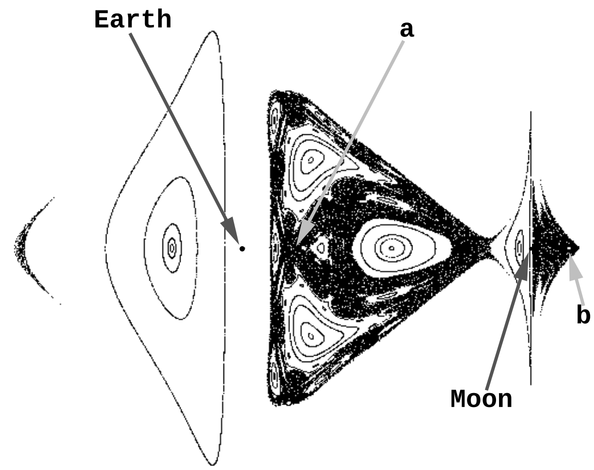


Figure 2: Phase space portrait of the Poincaré mapping of a $10^5$ iterate test orbit for the restricted three body problem with $J = -3.17948$. The point $(x, u)$ is plotted every time the flow pierces the surface $y = 0$ with positive $v$. The Earth and Moon are clearly labeled at their fixed locations in the rotating frame. The phase space locations of the starting point $\boldsymbol{a}$ near the Earth, and the target point $\boldsymbol{b}$ near a Moon orbit invariant torus are also labeled. Several invariant tori are also shown.

orbits may escape, so that we may have a long bounded test orbit. This we imagine is attained by an impulsive boost, $\Delta V$, of a spacecraft in a parking orbit around the earth to the energy $J_0$. FIG. 2. shows a phase space plot of a single "chaotic" test orbit with $10^5$ iterates. This test orbit may be stored as a "library" of known behaviors, and used for generating many missions in the accessible portion of phase space. Certain islands in phase space are inaccessible; these are bounded by invariant tori–some of which are shown. At the center of each island is a periodic orbit.

We choose the point $\boldsymbol{a} = (x_0, u_0)$ to achieve a fast chaotic orbit. A trial and error search for various $x_0$ near the Earth, but in the connected chaotic component that leads to the Moon, along the line segment $u_0 = 0$, gave the best results for an orbit at an altitude of 59669 km above the Earth's center. As our target, we choose the outermost invariant torus, marked "$b$" in FIG. 2., corresponding to a quasi-periodically precessing "ellipse" around the moon. As the actual target point, $\boldsymbol{b}$, we use the point of closest approach of our test orbit to $\boldsymbol{b}$, at an altitude of 13970 km above the Moon's center. From $\boldsymbol{b}$ a tiny perturbation will move the orbit onto the torus, thus achieving a state bound to the Moon without the large deceleration required by a Hohmann transfer. We define a "true" ballistic capture to the Moon (at constant energy) to be an orbit forward asymptotic to a Moon-orbiting invariant torus. This contrasts to a distinct definition by Belbruno [1]. We are searching for a Moon-ballistic capture in the sense of our strong definition.

The implication of solving Eq. (2), using the exact stable and unstable manifolds, is that near the pseudo-orbit we construct, there exists a true orbit which skips the recurrence. The orbit of $\boldsymbol{p}$ exactly yields the shadow orbit, by construction. When we use other curves to parameterize the variations, we lose this implication, but we gain another advantage. In constructing an Earth - Moon pseudo-orbit, even small variations along the stable and unstable manifolds in phase space imply variations in velocity *and position*. We wish to construct an orbit with only velocity errors, since teleportation is not physical, but rocket impulses are routine. According to the arguments of the previous paragraph, we may substitute the vector $(0, \delta u)$ for both $f_u$ and $f_s$ in Eq. (2) to find a real configuration space orbit, i.e., no position errors. With this choice, we find that $m = 12$, yielding a patch length $2m + 1 = 25$ steps, yields adequate recurrence error compression.

The $10^5$ iterate test orbit has a 10710 iterate segment which goes from $\boldsymbol{a}$ to $\boldsymbol{b}$. Fixing the recurrence distance to $\delta = 0.02$, we achieved a 58 iterate pseudo-orbit by cutting out 6 recurrence loops, and requiring a maximum perturbation of $\epsilon = 1.07 \times 10^{-4}$. Note that this implies perturbations to the real coordinates of $\delta u \leq 0.219$m/sec. The actual time along this orbit is $T = 172.3 = 2.05$ years.

Arbitrary $\Delta V$ maneuvers would change the value of the Jacobi constant, causing the rest of the pre-calculated orbit, constructed from segments of the constant energy orbit found in the stored library, to become invalid. However, by the implicit function theorem, Eq. (4) allows freedom to choose a $\delta v$ so as to conserve the value of $J$ under small variations $\delta u$ and $\delta x = \delta y = 0$. Thus we change the direction of the motion, by our maneuvers, and not the speed.

We show our chaotic orbit in the configuration space plot, FIG. 3. We can "see" the accelerating boosts of the Moon's gravitational pull as it swings by the earth orbiting spacecraft. These boosts perturb the spacecraft into just the right orientation to pass through the neck around $L_2$ exactly once with the correct speed and position so that it is captured by the Moon near the chosen invariant torus.

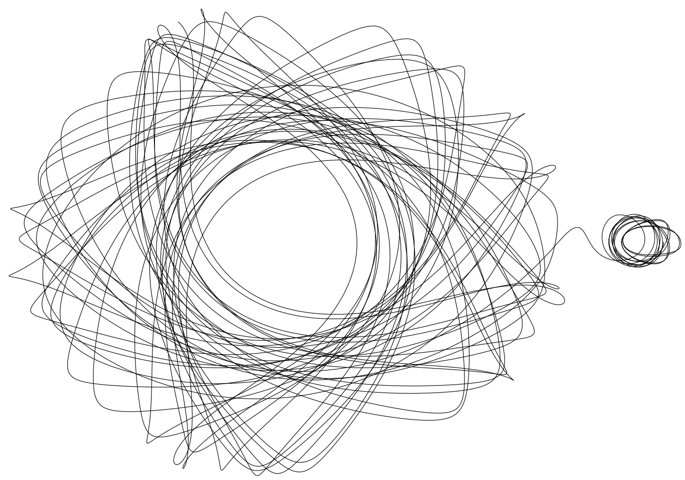

Figure 3: A configuration space plot $(x, y)$ of the 58 iterate chaotic transfer to the Moon. The final state at $\boldsymbol{b}$ is a precessing ellipse around the Moon corresponding to the targeted invariant torus.

The boosts required for our chaotic trajectory can be compared to those of a corresponding Hohmann like, two impulse transfer (the classic mission). Both orbits start at the (almost circular) parking orbit around the Earth at the starting altitude 59669 km with Jacobi constant $J = -7.1738$. An initial impulsive thrust is required for both transfers to increase the energy such that the zero velocity curves permit the transfer, $J > J_2$. The chaotic transfer requires an initial boost of $\Delta V = 744.4$ m/sec to attain $J_0 = -3.17948$. Additionally, it requires 4 patches with $\epsilon \leq 1.07 \cdot 10^{-4}$, and therefore the total change in velocity is bounded by $\Delta V \leq 6 \times 0.107$m/sec $= 0.659$m/sec. Finally, to jump from $\boldsymbol{b}$ to the targeted invariant torus requires $\epsilon = 4.363 \cdot 10^{-3}$ and therefore $\Delta V = 4.468$m/sec. Thus the total perturbation required by the chaotic transfer is 749.6m/sec.

In contrast, the Hohmann-like transfer requires an initial parallel burn of $\Delta V = 817.4$m/sec boosting the energy to $J = -2.761$. This gives a motion which is, roughly speaking (i.e. neglecting the effect of the moon ), a Kepler ellipse with apogee at $\boldsymbol{b}$. The spacecraft coasts until it arrives at $\boldsymbol{b}$, where a deceleration of $\Delta V = 402.5$ m/sec is applied. Therefore the total boost required for this Hohmann transfer is 1219.8 m/sec, but the transfer requires only 6.61 days.

Therefore we find that the ratio between the impulses is 0.615, or a 38%

advantage over the Hohmann orbit.

This is a significant improvement, but at the cost of a much longer (and circuitous) transfer. In terms of transferring passengers, the extra time is probably not worth the savings. However, for transferring freight, the $\Delta V$ savings of our orbit translates directly to a considerably smaller fuel requirement and therefore allows the transfer of a larger payload.

For example, suppose that a given booster is to be used in both cases, then an alternative figure of merit is given by the ratio of payload mass, $m_{pl}$ to propellant mass $m_{prop}$. This can be derived from the elementary rocket equation, which gives the ratio of final mass to initial mass: $m_f/m_o = e^{-\Delta V/gI_{sp}}$ where $I_{sp}$ is the specific impulse of the booster. For a chemical rocket $I_{sp} \approx 300$ sec. (450 sec is about the maximum achievable value with this technology). Using this value, and assuming that the structural mass of the booster is a fixed fraction, $\alpha = 15\%$, of the propellant mass, gives

$$\frac{m_{pl}}{m_{prop}} = \frac{1}{e^{\Delta V/gI_{sp}} - 1} - \alpha \ . \tag{5}$$

Then for our orbit $m_{pl}/m_{prop} = 3.30$ while the Hohmann transfer gives 1.80. Thus we are able to transfer 83% more payload from the circular orbit at **a** with the same booster.

Recently, another approach due to Belbruno was used to find chaotic transfer orbits to the moon utilizing the so-called "fuzzy boundary" [1]. This method was successfully applied to send the spacecraft Hiten to the Moon, thus saving an otherwise failed mission when the original Moon probe was lost. The Hiten orbit requires a restricted four body model, including the Sun, plus three configuration space directions. The technique is to send the spacecraft to the fuzzy boundary between the Earth and Sun, where their gravitational effects balance, so that only a small perturbation is necessary to reach the Moon in a "ballistic capture orbit" analagous to our orbit in that it requires almost no decelerating $\Delta V$. This orbit is much less circuitous than ours and requires approximately 4.6 months. However, a larger rocket burn is required to escape the Earth in order to reach the fuzzy boundary, well away from the Earth-Moon zero velocity curve at $J_c$. Our technique could also be applied to the restricted four body problem (with the added complication that the dimension of the phase space is increased since time cannot be eliminated by going to a rotating frame), and would give a systematic method for finding optimal orbits in this case as well.

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