Computing periodic orbits using the anti-integrable limit

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Abstract

Chaotic dynamics can be effectively studied by continuation from an anti-integrable limit. Using the Hénon map as an example, we obtain a simple analytical bound on the domain of existence of the horseshoe that is equivalent to the well-known bound of Devaney and Nitecki. We also reformulate the popular method for finding periodic orbits introduced by Biham and Wenzel. Near an anti-integrable limit, we show that this method is guaranteed to converge. This formulation puts the choice of symbolic dynamics, required for the algorithm, on a firm foundation.
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1 Introduction

The anti-integrable (AI) limit of a mapping is a singular limit in which the dynamics is not deterministic[1]. Each orbit in the AI limit is given by a sequence of symbols, and the dynamics becomes the unrestricted shift operator on the symbols. It is well known that when the AI limit is "nondegenerate" many of the AI states can be continued away from the limit, becoming orbits of the original mapping [2, 3]. This continuation theory, a consequence of the implicit function theorem, is much simpler than the corresponding continuation from an "integrable limit", which often requires the machinery of KAM theory.

Biham and Wenzel [4] introduced a technique for finding periodic orbits of maps, in particular maps that are obtained from a variational principle. They generalized the notion of "gradient search" to find the minimum of the variational function by introducing a "pseudo-gradient" system of differential equations. This is obtained by multiplying the gradient by a diagonal matrix of signs. The basic idea is that this will allow one to find critical points other than the minimum. This method has become popular [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] in spite of the fact that it has no rigorous foundation. Indeed, Grassberger et. al [18] found examples where the method fails to find unique orbits for certain parameter values in the Hénon map.

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In this paper we use the Hénon map as an example to illustrate the usefulness of the AI limit. At the AI limit, the Hénon map reduces to the full shift on two symbols, and we study orbits that continue from these states. We obtain an explicit bound on this continuation which happens to correspond exactly to the bound on the existence of the horseshoe obtained by Devaney and Nitecki [19] who used a geometrical argument. We then reformulate the Biham and Wenzel method and show that the pseudo-gradient system has a hyperbolic sink at the AI limit, and that this sink persists at least for the same parameter range for which we can continue the orbits.

2 Anti-Integrable Limit

The anti-integrable (AI) limit can be formulated in two ways. Consider a map, $f : M \to M$ on a d dimensional manifold M that depends upon some parameters. The general idea is to rewrite the system as an equivalent implicit relation $F : M \times M \to \mathbb{R}^d$ in such a way that F becomes singular when a parameter, say ϵ, is zero [1]. For example if $F(x, x') = \epsilon G(x, x') + H(x)$, then the "orbit" at $\epsilon = 0$ corresponds to any sequence of zeros of H–the dynamics is not deterministic. We say that $\epsilon = 0$ corresponds to the *anti-integrable limit* of the map f. If the derivative of H is nonsingular, then a straightforward implicit function argument can be used to show that (at least some of) the AI orbits can be continued for $\epsilon \neq 0$ to orbits of the map f [20, 3]. An AI limit with this property is called *nondegenerate*.

Maps that are derived from a Lagrangian variational principle often have an AI limit. For example, orbits of the Hénon map, written in the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y - k + x^2 \\ -bx \end{pmatrix} , \tag{1}$$

can be obtained as the critical point of an action,

$$W[\mathbf{x}] = \sum_t b^{-t} L(x_t, x_{t+1}) , \tag{2}$$

where we label time along the trajectory by a subscript t, and the factor b^{-t} allows the system to be non-area preserving. An orbit $\mathbf{x} = \{\ldots x_t, x_{t+1}, \ldots\}$ is a critical point of W, and $y_t = bL_2(x_{t-1}, x_t) = -L_1(x_t, x_{t+1})$. When the Lagrangian L can be put into the form $L(x, x') = \epsilon T(x, x') - V(x)$, then $\epsilon = 0$ corresponds to an AI limit [3]. This form is quite natural for "mechanical systems" where T represents kinetic and V potential energy.

For example, the Lagrangian for the Hénon map is

$$\hat{L}(x, x') \quad = \quad -xx' + \frac{1}{3}x^3 - kx . \tag{3}$$

Critical points of the action give the second order difference or Euler-Lagrange form of the map. To put (3) into a form appropriate for the AI limit, we rescale, defining

$$z = \epsilon x \quad , \quad \epsilon = \frac{1}{\sqrt{k}} ,$$

to obtain the new Lagrangian,

$$L(z,z') = \epsilon^3 \hat{L}(x,x') = -\epsilon z z' + \frac{z^3}{3} - z\,. \tag{4}$$

Critical points of the action correspond to sequences z_t such that

$$-\epsilon(z_{t+1} + b z_{t-1}) + z_t^2 - 1 = 0\,. \tag{5}$$

Of course this implicit system is easily solved for z_{t+1} whenever $\epsilon \neq 0$ to give an explicit dynamical system. However, when $\epsilon = 0$ (5) no longer defines a map, even though it still represents a critical point of W. This is the AI limit of the Hénon map. At this limit, "orbits" correspond to any bi-infinite sequence $\mathbf{z} \in \mathbb{S}$, where we denote the set of bi-infinite sequences with two symbols by

$$\mathbb{S} = \{\mathbf{s} : s_t \in \{1,-1\}\,, t \in \mathbb{Z}\}\,.$$

Each $\mathbf{s} \in \mathbb{S}$ continues to a unique orbit $\mathbf{z}(\epsilon)$ where $\mathbf{z}(0) = \mathbf{s}$ when ϵ is small enough [1, 3]. In fact, we can obtain an explicit upper bound on the range of ϵ for which this correspondence is guaranteed. It will be appropriate to use the l^∞ norm, $||\mathbf{x}||_\infty = \sup_t |x_t|$, and define B_M to be the closed ball of radius M around the point $\mathbf{s}$,

$$B_M(\mathbf{s}) = \{\mathbf{z} : ||\mathbf{z} - \mathbf{s}||_\infty \leq M\}\,. \tag{6}$$

With this notation, we can prove:

Theorem 1. *For every symbol sequence $\mathbf{s} \in \mathbb{S}$, there exists a corresponding unique orbit, $\mathbf{z}(\epsilon)$, of the Hénon map (5) such that $\mathbf{z}(0) = \mathbf{s}$ providing*

$$|\epsilon|(1+|b|) < \gamma_\infty \equiv 2\sqrt{1 - 2/\sqrt{5}} \approx 0.649839\,. \tag{7}$$

The orbit $\mathbf{z}(\epsilon)$ is contained in the ball $B_{M_\infty}(\mathbf{s})$ where

$$M_\infty = 1 - \sqrt{1 - \gamma\frac{\gamma + \sqrt{\gamma^2+4}}{2}}\,, \tag{8}$$

$$\gamma \equiv |\epsilon|(1+|b|)\,. \tag{9}$$

To prove the theorem, we will write orbits $\mathbf{z}(\epsilon)$ of the Hénon map as fixed points of an operator $\mathbf{T}$ whose t^{th} component is

$$T_t(\mathbf{z}) \equiv s_t\sqrt{1 + \epsilon(z_{t+1} + b z_{t-1})}\,, \tag{10}$$

where we choose the sign of the square root using a sequence $\mathbf{s} \in \mathbb{S}$. When $\epsilon = 0$, the operator becomes $\mathbf{T}(z) = \mathbf{s}$, which trivially has a unique fixed point corresponding to the AI state, $\mathbf{z} = \mathbf{s}$. We will use the contraction mapping theorem to show that for small enough ϵ the fixed point persists. The first step in the proof is to show that there is a domain $\mathcal{C}_1$ of the parameters (γ, M) such that $\mathbf{T}$ is a contraction on $B_M(\mathbf{s})$; this domain is illustrated

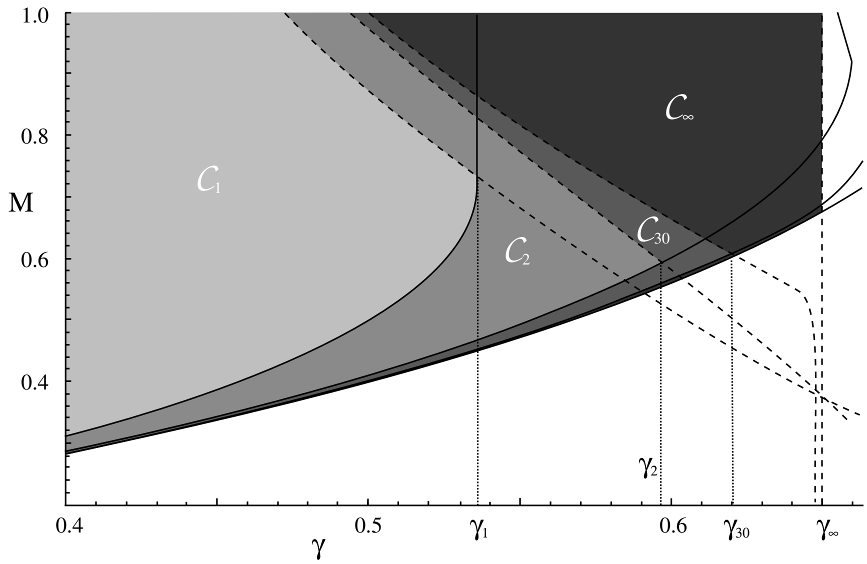

Figure 1: **Contraction regions $\mathcal{C}_n$ for $b = 1$ and $n = 1, 2, 30, \infty$.**

in Fig. 1. This is improved by bounding the domain $\mathcal{C}_n$ in the (γ, M) plane where $\mathbf{T}^n(\mathbf{z})$ is a contraction. It is easy to see that

$$\mathcal{C}_n \subseteq \mathcal{C}_{2n} ;$$

thus we can improve the bound on $\gamma(\epsilon)$ by iteration of $\mathbf{T}$. Figure 1 illustrates the parameter domains that we find for $n = 1, 2, 30$, and ∞. The maximal γ values for each of these iterates are denoted γ_n in Fig. 1.

To demonstrate that $\mathbf{T}^n$ is a contraction mapping we must show that for $(\gamma, M) \in \mathcal{C}_n$

1. $\mathbf{T}^n : B_M(\mathbf{s}) \to B_M(\mathbf{s})$
2. $||D\mathbf{T}^n(\mathbf{z})||_\infty < 1$ for all $\mathbf{z} \in B_M(\mathbf{s})$

The first requirement gives the solid curve in the figure defining a lower bound on M values in $\mathcal{C}_n$ as well as right boundary where the range of $\mathbf{T}^n$ is no longer real. The dashed boundary of $\mathcal{C}_n$ is given by the second requirement. The maximal value of γ is obtained in the limit $n \to \infty$.

We begin with a lemma to prove the first requirement.

Lemma 1.1. *For any $|\gamma| < 1/\sqrt{2}$, and any $M > M_\infty(\gamma)$ there is an N such that $\mathbf{T}^n$: $B_M(\mathbf{s}) \to B_M(\mathbf{s})$, $\forall n > N$.*

Proof:

Let $\mathbf{T}$ be defined by (10). For any $\mathbf{z} \in B_M(\mathbf{s})$ it is easy to see that

$$\alpha_n \leq ||\mathbf{T}^n(\mathbf{z})||_\infty \leq \beta_n , \tag{11}$$

where the sequences α_n and β_n are given by the iterations

$$\begin{aligned}\beta_{n+1} &= f(\beta_n) \equiv \sqrt{1+\gamma\beta_n}\,, \\ \alpha_{n+1} &= \sqrt{1-\gamma\beta_{n+1}}\,,\end{aligned}$$

with the initial conditions $\beta_0 = 1 + M$ and $\alpha_0 = 1 - M$. The map $f(\beta)$ has a single attracting fixed point

$$\beta_\infty = \frac{\gamma + \sqrt{\gamma^2+4}}{2}\,.$$

Each of the α_n must be real, so we must have $1-\gamma\beta_n \geq 0$. This requirement gives a right boundary to the region in the (γ, M) plane where $\mathbf{T}^n$ exists. As $n \to \infty$ these boundaries approach the vertical line defined by

$$1 - \gamma\beta_\infty = 0 \Rightarrow \gamma = 1/\sqrt{2}\,, \tag{12}$$

which gives one of the bounds in the lemma.

Finally, (11) implies that

$$||\mathbf{T}^n - \mathbf{s}||_\infty \leq \max\left(|\alpha_n - 1|, |\beta_n - 1|\right) = 1 - \alpha_n\,.$$

Thus the requirement that $\mathbf{T}^n$ maps $B_M(\mathbf{s})$ into itself gives the implicit relation $1-\alpha_n \leq M$. As n approaches infinity, these domains approach the region

$$M \geq M_\infty(\gamma) = 1 - \alpha_\infty\,.$$

For each γ the sequence M_n for which $1-\alpha_n-M_n = 0$ converges monotonically to M_∞ from above; therefore for any $M > M_\infty(\gamma)$, there is an N such that $\mathbf{T}^n : B_M(\mathbf{s}) \to B_M(\mathbf{s})\,, \forall n > N$. □

Proof of theorem (1) Let $B_M(\mathbf{s})$ be defined by (6) and $\mathbf{T}$ by (10). The map $\mathbf{T}^n$ is a contraction if $||D\mathbf{T}^n(z)||_\infty < 1$ for all $\mathbf{z} \in B_M(\mathbf{s})$. Using the chain rule gives

$$||D\mathbf{T}^n||_\infty \leq \frac{\gamma^n}{2^n \prod_{j=1}^n \alpha_j} \tag{13}$$

From this and lemma 1.1, the map $\mathbf{T}^n$ is a contraction in the region $\mathcal{C}_n$ given by

$$\mathcal{C}_n \equiv \{(\gamma, M) : 1-\alpha_n \leq M\,,\ \gamma \leq 2\prod_{j=1}^{n} \alpha_j^{1/n}\,,\ \alpha_j \text{ Real for } j \leq n\}\,. \tag{14}$$

We find the boundary of $\mathcal{C}_\infty$ by noting that the product in (14) converges to α_∞ because the sequence α_j converges geometrically to the fixed point. Thus there exists an N such that the map $\mathbf{T}^n$ is a contraction for all $n > N$ whenever

$$\gamma < 2\alpha_\infty\,. \tag{15}$$

Using the form for α_∞ we obtain the bound

$$\gamma^2 < 4(1 - 2/\sqrt{5}) , \tag{16}$$

which directly gives (7). This bound is clearly more restrictive than (12) so we can conclude that for any

$$(\gamma, M) \in \mathcal{C}_\infty = \{(\gamma, M) : M > M_\infty(\gamma), \gamma < \gamma_\infty\}$$

there is an N such that $\forall n > N$, the map $\mathbf{T}^n(\mathbf{z})$ is a contraction map whenever $\mathbf{z} \in B_M(\mathbf{s})$ The Banach fixed point theorem then implies that $\mathbf{T}^n$ has a unique fixed point in $B_{M_\infty}(\mathbf{s})$ for all $n > N$. But if $\mathbf{z}$ is a fixed point both of $\mathbf{T}^{N+1}(\mathbf{z})$ and of $\mathbf{T}^{N+2}(\mathbf{z})$ then it must be a fixed point of $\mathbf{T}$. Since this is true for an arbitrary symbol sequence, for any γ in the range (16), there is a corresponding unique orbit of the Hénon map. □

By a completely different argument, Devaney and Nitecki[19] proved that the non-wandering set of the Hénon map is hyperbolic and conjugate to the 2-shift for exactly the same range of γ that we give in Theorem 1.

3 The Method of Biham and Wenzel

Biham and Wenzel [4, 5] introduced a popular algorithm for finding periodic orbits of maps. They present the method in four steps

1. Write the map in variational form with action $W[\mathbf{x}]$. Critical points of W give the desired orbits.

2. Use symmetries or some other technique to label the various critical points of W symbolically with a symbol sequence of signs, s_j.

3. Choose an "arbitrary" initial guess for a periodic orbit, $x_j(0), j = 0, 1, \ldots n-1$ of period n.

4. Introduce a pseudo-gradient dynamics with a fictitious time τ to find an extremum of the action

$$\frac{d\mathbf{x}}{d\tau} = -\mathrm{Diag}(\mathbf{s})\nabla W[\mathbf{x}] . \tag{17}$$

   For a period n orbit, we set $x_{j+n}(\tau) = x_j(\tau)$, and solve only the differential equations for $x_0, x_1, \ldots x_{n-1}$.

Note that when $\mathrm{Diag}(\mathbf{s}) = \mathbf{I}$ this method is a standard gradient search for finding a local minimum of W, and if we set $\mathrm{Diag}(\mathbf{s}) = -\mathbf{I}$, then we would simply be finding a local maximum of W. More generally, the symbols are chosen in an attempt to "identify the critical points" of the action, but there is "no general way to find" the symbolic representation [5].

The hope is that for each choice of s_j (and some range of map parameter) the system (17) either has a "unique" attracting fixed point, or else the "corresponding" periodic orbit does not exist. This claim seems at least plausible for the Hénon map, since it has at most 2^n fixed points of period n, and there are precisely 2^n choices of periodic sequences $\mathbf{s}$. However, Grassberger et. al [18] have found examples for the Hénon map where the method fails to find unique orbits at certain parameter values. More generally, there is no guarantee that the fixed point of (17) corresponding to the orbit of interest is attracting. Moreover, this system of equations certainly can have other attractors, including chaotic ones.

A rigorous foundation for the Biham and Wenzel method can be given when the map has an AI limit. Suppose the discrete Lagrangian has the form $L(x, x') = \epsilon T(x, x') - V(x)$, and that the potential V has a discrete set of nondegenerate critical points $\mathrm{Crit}(V) = \{c_0, c_1, c_2, \ldots c_m, c_{m+1}\}$, where we order the points, $c_j < c_{j+1}$ with the convention that $c_0 = -\infty$ and $c_{m+1} = \infty$. When $\epsilon = 0$ (17) decouples, and becomes simply

$$\frac{dx_j}{dt} = -s_j V'(x_j) \,.$$

Of course the equilibria of these differential equations are precisely the AI states, i.e., any sequence of critical points, $x_j = c_{k_j}$, $k_j \in \{1, 2, \ldots m\}, j \in \mathbb{Z}$. Such an equilibrium is a sink if $s_j = sgn(V''(c_{k_j}))$. For example when the c_k are minima of V, then $s_j = 1$ as we expect. The basin of attraction of the sink is the box

$$D_{\mathbf{s}}(0) = \{\mathbf{x} : c_{k_j - 1} < x_j < c_{k_j + 1}\} \,.$$

Since hyperbolic fixed points of a vector field are preserved under a C^1 perturbation [21], we immediately have the theorem

Theorem 2. *If $L = \epsilon T(x, x') - V(x) \in C^2$, and the critical points $\{c_k\}$ of V are nondegenerate, then there is an ϵ_{max} such that for all $|\epsilon| < \epsilon_{max}$ and all sequences of critical points $\{c_{k_j}, j \in \mathbb{Z}\}$, the system of differential equations (17), with $s_j = sgn(V''(c_{k_j}))$, has a hyperbolic sink, $\mathbf{x}^*$, that continues from $x_j = c_{k_j}$ at $\epsilon = 0$.*

This reformulation of the Biham-Wenzel method points out several important things. First, the correct choice of the signs is determined by the signature of the potential V at the appropriate critical point. In fact, as originally formulated, the method cannot possibly find all orbits when V has more than two critical points. Rather, an orbit is determined first by the choice of anti-integrable state c_{k_j}; the sequence of signs s_j is determined subsequently. Finally, as we will explicitly demonstrate below, it is sensible to use the AI state as the initial condition for the method.

For the Hénon map, we use the scaled Lagrangian to obtain the system

$$\frac{dz_j}{d\tau} = s_j \left(\epsilon(z_{j+1} + b z_{j-1}) - z_j^2 + 1\right) \,. \tag{18}$$

When $\epsilon = 0$, the differential equations decouple and give the simple equations

$$\frac{dz_j}{d\tau} = s_j \left(1 - z_j^2\right) \,,$$

for which there is a unique attracting fixed point at $z_j = s_j$ with a basin of attraction

$$D_{\mathbf{s}}(0) = \{\mathbf{z} : -1 < z_j < \infty\} \,.$$

We can use theorem (1) to obtain a bound on ϵ for the persistence of the hyperbolic fixed points of (18). We know from this theorem that there is a fixed point $\mathbf{z}^*$ for each $\mathbf{s}$ when $\gamma < \gamma_\infty$. The linearization of (18) around this fixed point is

$$\begin{aligned} \frac{d\zeta_i}{d\tau} &= \sum_{j\in\mathbb{Z}} A_{ij}\zeta_j \,, \\ A_{ij} &= s_i\left[\epsilon(\delta_{i+1,j} + b\delta_{i-1,j}) - 2z_j^*\delta_{i,j}\right] . \end{aligned} \tag{19}$$

By the Gerschgorin circle theorem, the eigenvalues λ_k of $\mathbf{A}$ are contained within the union of the disks centered at the diagonal elements with radius given by the sum of the magnitudes of the off-diagonal elements [22]. Since the row sum is bounded by $|\epsilon|(1+|b|) = \gamma$, we obtain

$$\lambda_k \in \bigcup_j \{\lambda : \left|\lambda + 2|z_j^*|\right| \leq \gamma\} \,.$$

According to theorem (1), $\mathbf{z}^* \in B_{M_\infty}(\mathbf{s})$, where M_∞ is given by (8); therefore,

$$Re(\lambda) \leq -2(1 - M_\infty) + \gamma \,. \tag{20}$$

This is negative precisely when $\gamma < \gamma_\infty$. Thus we have proven

Theorem 3. *When $|\epsilon|(1 + |b|) < \gamma_\infty$, then the orbit of the Hénon map labeled by the AI symbol sequence $\mathbf{s}$ is a hyperbolic sink for the system (18).*

The basin of attraction of the sink initially includes the AI state $\mathbf{z} = \mathbf{s}$. We will next show that this remains true providing ϵ is small enough. Thus the proper initial condition to use for the Biham-Wenzel equations is the AI state.

Theorem 4. *The AI sequence $\mathbf{s}$ is in the basin of attraction for the fixed point $\mathbf{z}^*$ of (18) providing*

$$|\epsilon|(1 + |b|) < 0.555668 \,.$$

Proof: Suppose $\mathbf{z}^*$ is an orbit of the Hénon map for some fixed $\gamma < \gamma_\infty$. Using (18), the deviation $\zeta = \mathbf{z} - \mathbf{z}^*$ from this orbit satisfies the system of equations

$$\frac{d\zeta_i}{d\tau} = \sum_{j\in\mathbb{Z}} A_{ij}\zeta_j - s_i\zeta_i^2 \,, \tag{21}$$

where $\mathbf{A}$ is given in (19). It is easy to see that

$$\frac{d}{d\tau}||\zeta||_\infty \leq (-2||\mathbf{z}^*||_\infty + \gamma + ||\zeta||_\infty)||\zeta||_\infty \,,$$

so $||\zeta||_\infty$ decreases when

$$0 < ||\zeta||_\infty < 2(1 - M) - \gamma \,,$$

This implies that the basin of attraction of the sink contains this ball:

$$D_{\mathbf{s}}(\gamma) \supset \{\mathbf{z} : ||\mathbf{z} - \mathbf{z}^*|| < 2(1 - M) - \gamma\}$$

If the AI state $\mathbf{s}$ is to be in this basin, then we require that $\mathbf{s} \in D_{\mathbf{s}}(\gamma)$, but since we know that $||\mathbf{z}^* - \mathbf{s}||_\infty \leq M_\infty$, this is true when

$$3M_\infty(\gamma) < 2 - \gamma \Rightarrow \gamma < 0.55566\,.$$

□

4 Conclusions

A large class of dynamical systems can be formulated to have a nondegenerate AI limit. In this case a simple contraction mapping argument can be used to show that many of the symbolic orbits at the AI limit persist away from the limit. We have applied these ideas to the Hénon map to show that all possible bounded orbits exist in the range (7). Translating this back to the original parameters k and b of the map (1) gives

$$k > \frac{5 + 2\sqrt{5}}{4}(1 + |b|)^2$$

which is exactly the same bound as that found by Devaney and Nitecki [19] using a geometrical argument for the existence of a Smale horseshoe. The advantage of the AI argument is that it can be easily generalized to systems where the geometrical argument might be difficult, such as higher dimensional maps. Moreover, the AI theory can also be used to give bounds for the existence of orbits corresponding to various subshifts of finite type [23].

Our main interest in the AI limit is to use it as the basis for a numerical method to find orbits of various types. One such method we discussed here is the "pseudo-gradient" algorithm of Biham and Wenzel. We showed that this method, which was previously only justified heuristically, has a rigorous foundation close enough to an AI limit. Indeed, the signs in the matrix Diag($\mathbf{s}$) are determined by signature of the critical point at the AI limit. We also were able to show that the AI state can be used as an initial condition for the method. However, our theorem applies only for a limited range of parameter values, and one might reasonably apply the method over a wider range of parameters.

Some care is advisable however, since it is known that the method of Biham and Wenzel can fail far from the AI limit [18]. We will show in a forthcoming paper [23] that in some applications, such as that in [6], the method does appear to find all orbits.

Rather than use the Biham and Wenzel method we implement a simple "continuation" technique for our numerical studies [23]. This method is guaranteed to find all bounded orbits of the map that extend to the AI limit (there could be "bubbles" that do not extend, but as far as we know these have never been seen for the Hénon case). For example, in the area preserving case ($b = 1$), our numerical results indicate that the horseshoe is destroyed at $\epsilon = 0.41887923$, when a pair of orbits homoclinic to the fixed point, $\mathbf{s} = \{\ldots, 1, 1, \ldots\} \equiv \{+^\infty\}$, collide in a saddle-node bifurcation. These orbits have the symbol sequences

$$\{+^\infty - - - +^\infty\} \Longleftrightarrow \{+^\infty - + - +^\infty\}\,.$$

This parameter value also corresponds to the accumulation point of an infinite number of periodic saddle-node bifurcations, and hence the first value at which the bounded orbits of the Hénon map have nonzero measure.

References

[1] S. J. Aubry. Anti-integrability in dynamical and variational problems. *Physica D*, 86:284–296, 1995.

[2] S. Aubry and G. Abramovici. Chaotic trajectories in the standard map, the concept of anti-integrability. *Physica D*, 43:199–219, 1990.

[3] R.S. MacKay and J.D. Meiss. Cantori for symplectic maps near the anti-integrable limit. *Nonlinearity*, 5:149–160, 1992.

[4] O. Biham and W. Wenzel. Characterization of unstable periodic orbits in chaotic attractors and repellers. *Phys. Rev. Lett.*, 63:819–822, 1989.

[5] O. Biham and W. Wenzel. Unstable periodic orbits and symbolic dynamics of the complex Hénon map. *Phys. Rev. A*, 42:4639–4646, 1990.

[6] M.J. Davis, R.S. MacKay, and A. Sannami. Markov shifs in the Hénon family. *Physica D*, 52:171–178, 1991.

[7] W. Wenzel, O. Biham, and C. Jayaprakash. Periodic orbits in the dissipative standard map. *Phys. Rev. A*, 42:6550–6557, 1991.

[8] O. Biham and M. Kvale. Unstable periodic orbits in the stadium billiard. *Phys. Rev. A*, 46:6334–6339, 1992.

[9] B. Dey. Unstable periodic orbits and characterization of spatial chaos in a nonlinear monatomic chain at the T=0 first-order phase transition point. *Phys. Rev. B*, 52:220–224, 1995.

[10] G. D'Alessandro, P. Grassberger, S. Isola, and A. Politi. On the topology of the Hénon map. *J. Phys. A*, 23:5285–5294, 1990.

[11] K.T. Hansen. Remarks on the symbolic dynamics for the Hénon map. *Phys. Lett. A*, 165:100–104, 1992.

[12] A. Politi and A. Torcini. Towards a statistical mechanics of spatiotemporal chaos. *Phys. Rev. Lett.*, 69:3421–1324, 1992.

[13] B. Hunt and E. Ott. Optimal periodic orbits of chaotic systems occur at low period. *Phys. Rev. E*, 54:328–337, 1996.

[14] R. Skodje and M. Davis. Statistical rate theory for transient chemical species: Classical lifetimes for periodic orbits. *Chem. Phys. Lett.*, 175:92–100, 1990.

[15] H.P. Fang. Dynamics for a two-dimensional antisymmetric map. *J. Phys. A*, 27:5187–5200, 1994.

[16] H. Kaplan. Type I intermittancy for the Hénon family. *Phys. Rev. E*, 48:1655–1669, 1993.

[17] Y. Nagai and Ying-Cheng Lai. Characterization of a blowout bifurcation by unstable periodic orbits. *Phys. Rev. E*, 55:R1251–R1254, 1997.

[18] P. Grassberger, H. Kantz, and U. Moenig. On the symbolic dynamics of the Hénon map. *J Phys A*, 22(24):5217–5230, 1989.

[19] R.L. Devaney and Z. Nitecki. Shift automorphisms in the Hénon mapping. *Commun. Math. Phys.*, 67:137–146, 1979.

[20] S. J. Aubry. The concept of anti-integrability: Definition, theorems and applications to the standard map. *Twist Mappings and their Applications, Ed Richard McGehee, Kenneth R. Meyer*, pages 7–54, 1992.

[21] M.W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems and Linear Algebra.* Academic Press, 1974.

[22] J. Stoer and R. Bulirsch. *Introduction to Numerical Analysis.* Springer-Verlag, 1980.

[23] D. Sterling and J.D. Meiss. Homoclinic bifurcations for the area-preserving Hénon map. *In Progress*, 1998.