A linear system and explicit solutions for approximate linear phase filters

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Abstract

Filters with exact or approximate linear phase response around the origin have been studied for many years and applied to a variety of problems. In practice one may need filters satisfying other properties besides linear phase. These additional properties may prevent exact linear phase and impose restrictions on the order of approximation. Since the order of linear phase approximation depends on the number of vanishing Taylor coefficients in the phase expansion around zero, these conditions have been thought to impose non-linear conditions on the filter coefficients.

It is shown here that any order of linear phase approximation is equivalent to a set of linear conditions on the coefficients of the filter. The linear system is defined from a set of conditions on the filter frequency response, and these conditions are the same whether the filter is FIR, IIR (rational or non-rational), analog or digital. In the latter case, the conditions correspond to vanishing shifted odd moments of the filter coefficients. The shift is an arbitrary real number which equals the DC group delay.

Simultaneous phase and amplitude approximation can be also shown to lead to a linear system. For digital filters, the flatness of the amplitude response around the origin is equivalent to vanishing shifted even moments of the coefficients.

Explicit expressions for digital FIR filters with optimal phase approximation or with optimal simultaneous amplitude and phase approximation of an ideal response are derived. In both cases, nonoptimal approximations are expressed as linear combinations of the optimal filters. By identifying all possible solutions, one obtains the basic building blocks present in a variety of filter designs.

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I Introduction

Filters with linear phase response are of considerable importance for a variety of applications. Unfortunately, other desirable properties of the filters could be incompatible with the linear phase property. For example, except for the Haar filter, there are no FIR perfect reconstruction filters with *exact* linear phase [11, 2], even though it is still possible to design examples if one asks for nearly linear phase [8]. These filters are named coiflets [3] and, as shown in [8], their construction is greatly simplified by an appropriate linear change of variables involving the DC group delay. A similar approach is followed here, and the delay is always view as an extra parameter that can take any real value.

It is shown here that approximate linear phase (ALP) around the origin can be characterized by simple, linear conditions on the coefficients of the filter. In fact, the result follows from an abstract property of functions and can be applied to a variety of filters provided they have real coefficients.

The ALP system is explicitly solved for digital FIR filters and we obtain explicit expressions for maximally flat delay filters of any length and any DC group delay. These filters are particular cases of hypergeometric functions and can be associated with a variety of special functions. Thus, recurrences, location of zeroes, integral representations, and many other properties are available.

In agreement with theoretical results, these optimal ALP filters coincide with those derived in other constructions. For a range of delay values, these optimal solutions can be obtained by appropriate transformations of Abele's maximally flat distributed linear phase filters (See [1], [4] or [10, Section 6.3].) They also can be directly obtained from Thiran's all-pole digital filters with maximally flat delay [12] by reversing the delay sign and multiplying by an appropriate constant. However, the proof presented here is more general and covers all possible values of the delay even those values leading to a singular system. The solutions of these singular systems are also fully described. These solutions include symmetric polynomials, that is, filters with exact linear phase. In this way, exact linear phase is described as a particular case of the ALP system that only arises for integer or half-integer choices of the delay but is nevertheless naturally integrated into the general framework of ALP filters.

With respect to simultaneous amplitude and phase approximation, it will be shown that if the order of amplitude approximation is at most twice the ALP order then the amplitude approximation conditions are also linear conditions on the coefficients of the filter. Using this result one can easily derive the well known optimal FIR approximation of an ideal fractional delay filter (See [7] and references therein.) These optimal filters are also hypergeometric functions.

For clarity, we present first a brief summary of a program to approach other linear phase designs.

IIR filters Many IIR descriptions can be obtained using those for the FIR case. For example, for digital filters $H = P/Q$ where P and Q are FIR filters, any order of linear phase approximation for $H(z)$ is equivalent to the same order for $P(z)Q(z^{-1})$.

Analog filters Distributed filters with any order of ALP can be obtained by the standard bilinear transformation applied to a digital filter with the same order of approximation. Since FIR filters are transformed into IIR filters, the distributed case can be obtained once we know the appropriate solutions for both FIR and IIR digital filters. The optimal lumped FIR filters are Bessel polynomials. They can be used to obtain other FIR or IIR filters.

Nonoptimal filters A filter with any order of linear phase approximation can be expressed as linear combinations of optimal ALP filters. The constants in the linear combination are free parameters that can be used to impose additional properties.

Simultaneous Amplitude and Phase Approximation The optimal FIR filters approximating an ideal fractional delay can be used to describe other simultaneous approximations where the order of approximation of the amplitude and phase differ.

We now point out some of the advantages of the program presented.

- The linear formulation and the recognition of common properties for all ALP filters yields a general framework for the study of linear phase properties.
- Optimal ALP filters are of interest in their own right but their properties are also important for the design of all other ALP filters. Reciprocally, known constructions can be recast in terms of these filters and previous results can be used to further understand the properties and structure of the optimal cases.
- The previous points also apply to simultaneous amplitude and phase approximation provided that the order of amplitude approximation is at most twice the order of phase approximation.

In this paper we focus on deriving the properties common to all ALP filters and its consequences for digital FIR filters. Other filter designs will be discussed elsewhere.

The summary of the paper is a follows. The conditions for ALP and for simultaneous amplitude and phase approximation are presented in Section II. In Section III these general results are specialized to digital filters and necessary conditions for the filter magnitude to be less than unity are derived in terms of the delay. The linear system for ALP around an arbitrary frequency is also presented. In Section IV all ALP FIR digital filters are described as linear combinations of maximally flat delay filters for which explicit expressions are derived. Integer or half-integer choices of the delay lead to a representation involving maximal and symmetric polynomials. Two different examples are presented in Section V. Simultaneous amplitude and phase approximation filters are described as linear combinations of explicit optimal filters which approximate an ideal fractional delay. In the second example we discuss some properties of maximally flat delay filters.

Notation

D denotes the derivative operator and xD the operator $x\frac{d}{dx}$. For any operator T, its n-th iteration is denoted T^n where T^0 is the identity operator. We assume enough derivatives for all functions under consideration.

The set of real numbers is denoted by $\mathbf{R}$ and the set of integers by $\mathbf{Z}$. $\mathbf{R}_N[X]$ is the set of polynomials with real coefficients and degree (deg) less or equal than N. Polynomials always occur in positive powers of the variable. We will use the set of $\mathcal{R}_N$ of *symmetric* polynomials,

$$\mathcal{R}_N = \{P \in \mathbf{R}_N[X] : z^N P(\frac{1}{z}) = P(z)\}$$
$$\mathcal{R}_{\{N\}} = \{P \in \mathcal{R}_N : \deg(P) = N \quad \text{and} \quad P(1) = 1\}.$$

For example, z^4 is symmetric because belongs to $\mathcal{R}_8$ even though it does not belong to $\mathcal{R}_4$ or $\mathcal{R}_{\{8\}}$.

$F(a, b; c; z)$ is the Gauss hypergeometric series of upper parameters a and b, lower parameter c, and argument z.

By a filter we always imply its z-transform. We only consider real low-pass filters and thus a FIR filter is a polynomial $\sum_{k=0}^{N} h_k z^k$, with real coefficients $\{h_k\}$ and $\sum_k h_k = 1$.

The factorial powers and generalized binomial coefficients are defined for any complex z and any nonnegative integer n as $z^{\underline{0}} = 1$,

$$z^{\underline{n}} = z(z-1)\cdots(z-(n-1)),$$

and $\binom{z}{n} = z^{\underline{n}}/n!$.

For $h > 0$ we use the notation $[a : b : h] = \{a + kh : k \in \mathbf{Z} \quad \text{and} \quad 0 \leq k \leq \frac{b-a}{h}\}$. We use parenthesis instead of brackets to exclude endpoints.

The symbol δ_{nm} is defined as $\delta_{nm} = 1$ if $m = n$, and $\delta_{nm} = 0$ otherwise. $\overline{z}$ denotes the complex conjugate of the complex number z.

II Conditions for approximate linear phase and for simultaneous amplitude and phase approximation

We first present an informal approach to the conditions for ALP. Write,

$$e^{-j\gamma\xi}H(e^{j\xi}) = \sum_{k=0}^{\infty} h_k e^{j\xi(k-\gamma)} = a(\xi)e^{j(p(\xi)-\gamma\xi)}, \tag{1}$$

where the real functions a and p are the amplitude and phase response of the filter H and γ is the DC group delay.

For a function f, the Taylor expansion of $f(e^x)$ is

$$f(e^x) = \sum_{n=0}^{\infty} \frac{(xD)^n f(1)}{n!} x^n.$$

Thus, for $f(x) = x^{-\gamma}H(x)$ at $x = j\xi$,

$$\begin{aligned} e^{-j\gamma\xi}H(e^{j\xi}) &= \sum_{n=0}^{\infty} \frac{(xD)^n(x^{-\gamma}H(x))(1)}{n!}(j\xi)^n \\ &= \sum_{n=0}^{\infty} \mathcal{M}_{2n}(-1)^n\xi^{2n} + j\sum_{n=0}^{\infty} \mathcal{M}_{2n+1}(-1)^n\xi^{2n+1}, \end{aligned} \tag{2}$$

where the real numbers

$$\mathcal{M}_n = \frac{1}{n!}\sum_k (k-\gamma)^n h_k, \tag{3}$$

are the shifted moments of the sequence $\{h_k\}$.

From (1), for H to be ALP, we expect the function $e^{-j\gamma\xi}H(e^{j\xi})$ to be close to a real function. For its imaginary part to vanish up to a certain order, Eq. (2) indicates that some odd moments should vanish. We will show that this is indeed the case and that $\mathcal{M}_{2n+1} = 0$ for $0 \le n < N$ is actually equivalent to $D^{2n+1}(p(\xi) - \gamma\xi)(0) = 0$ for $0 \le n < N$.

If the filter is ALP, the first term of the sum in (2) should somehow approximate the amplitude response. This intuition is again correct and, as it is shown in Corollary 3, flat amplitude around zero is equivalent with vanishing even moments.

Since we would like to apply our results to real filters H whose frequency responses can take the form $H(e^{j\xi})$, $H(j\xi)$, or $H(j\tan\xi)$, we consider complex valued functions $f(\xi)$ such that $f(-\xi) = \overline{f(\xi)}$. That is, we ask the real and imaginary parts of f to have even and odd symmetry. With the low-pass condition $f(0) = 1$, we can write $f(\xi) = a(\xi)e^{jp(\xi)}$, where the *amplitude* $a(\xi)$ is an even real function and the *phase* $p(\xi)$ is an odd real function.

Our first result is simple to prove but it has far reaching consequences.

Theorem 1 *Let $f(\xi)$ be a function that takes complex values, and such that $f(-\xi) = \overline{f(\xi)}$ and $f(0) = 1$. Consider its representation in a neighborhood of $\xi = 0$,*

$$f(\xi) = a(\xi)e^{jp(\xi)}, \tag{4}$$

where a is an even and p an odd function. For γ a real number and for all integers n, $0 \le n < N$ the following conditions are equivalent

$$D^{2n+1}p(0) = \gamma\delta_{n0}, \quad \textit{and} \tag{5}$$

$$D^{2n+1}(e^{-j\gamma\xi}f(\xi))(0) = 0. \tag{6}$$

Consequently,

$$p(\omega) = \gamma\omega + o(\omega^{2N}) \quad \textit{as} \quad \omega \to 0. \tag{7}$$

Proof Let $F(\xi) = e^{-j\gamma\xi}f(\xi)$. From (4),

$$\ln(F(\xi)) = \ln(a(\xi)) + j(p(\xi) - \gamma\xi). \tag{8}$$

Using that $a(\xi)$ is an even function,

$$D^{2n+1}(\ln F)(0) = jD^{2n+1}(p(\xi) - \gamma\xi)(0).$$

The result follows from the lemma in Appendix A because $\ln'(F(0)) \neq 0$. ∎

For the magnitude of f in (4) to be flat around zero, we need the derivatives of the function a to vanish at zero. When f has ALP, the next theorem implies that simultaneous phase and amplitude approximation is equivalent to vanishing even derivatives of $e^{-j\gamma\xi}f(\xi)$ at $\xi = 0$.

Theorem 2 *Let f, a, and p as in (4), and N and M any positive integers.*

1. *If $D^{2n+1}p(0) = \gamma\delta_{n0}$ for $0 \le n < N$, then*

$$D^{2n}a(0) = D^{2n}(e^{-j\gamma\xi}f(\xi))(0) \quad \textit{for} \quad 0 \le n \le N. \tag{9}$$

2. *If $D^n(e^{-j\gamma\xi}f(\xi))(0) = \gamma\delta_{n0}$ for $0 \le n < M$, then*

$$D^n(e^{-j\gamma\xi}f(\xi))(0) = D^n a(0) + jD^n(p(\xi) - \gamma\xi)(0) \quad \textit{for} \quad 0 \le n < 2M. \tag{10}$$

Proof Write $a(\xi) = F(\xi)G(\xi)$, where $F(\xi) = e^{-j\gamma\xi}f(\xi)$ and, because of the condition on the phase p,

$$G(\xi) = e^{j\xi^{2N+1}u(\xi^2)},$$

for some function u.

We have $D^kG(0) = \delta_{k0} \quad \text{for} \quad 0 \le k \le 2N$, and thus the first part follows because for $0 \le n \le N$,

$$\frac{D^{2n}}{2n!}a(0) = \sum_{k=0}^{2n} \frac{D^{2n-k}}{(2n-k)!}F(0)\frac{D^k}{k!}G(0) = \frac{D^{2n}}{2n!}F(0).$$

For the second part, write $M = 2N + \delta$, where δ is 0 or 1. By Theorem 1,

$$D^{2n+1}p(0) = \gamma\delta_{n0} \quad \text{for} \quad 0 \le n < N,$$

and the previous part implies $D^{2n}a(0) = \delta_{n0} \quad \text{for} \quad 0 \le n \le N$ or $D^n a(0) = \delta_{n0} \quad \text{for} \quad 0 \le n < M$, because a is an even function.

We obtain the result taking derivatives in (8) and applying the following consequence of (24) to $g(x) = \ln x$ and $u(x) = F(x)$ or $u(x) = a(x)$.

If $D^k u(a) = \delta_{nk} \quad \textit{for} \quad 0 \le k < M$, *then* $D^n(g \circ u)(a) = D^n u(a)\, Dg(u(a)) \quad \textit{for} \quad 0 < n < 2M.$ ■

III Conditions for digital filters

Let $f(\xi) = H(e^{j\xi})$ be the frequency response of a digital filter H. We have,

$$\frac{D^n}{n!}(e^{-j\gamma\xi}f(\xi))(0) = \frac{j^n}{n!}(xD)^n(x^{-\gamma}H(x))(1) = j^n\mathcal{M}_n, \tag{11}$$

where $\mathcal{M}_n$ are the moments defined in (3). We now summarize the relationship between these shifted moments and the derivatives at zero of the functions a and p.

Corollary 3 *Let* H *be a function with Laurent expansion on the unit circle,* $H(z) = \sum_{k\in\mathbf{Z}} h_k z^k$, *where* $\{h_k\}$ *are real,* $H(1) = 1$, *and* $H(e^{j\xi}) = a(\xi)e^{jp(\xi)}$ *where* a *and* p *are real functions,* a *even and* p *odd.*

(Phase) *For* $0 \le n < N$ *the following conditions are equivalent,*

$$\mathcal{M}_{2n+1} = 0 \quad \textit{and}$$
$$D^{2n+1}p(0) = \gamma\delta_{n0}.$$

(Amplitude) *If* $\mathcal{M}_{2n+1} = 0$ *for* $0 \le n < N$ *then*

$$\frac{D^{2n}}{(2n)!}a(0) = \mathcal{M}_{2n} \quad \textit{for} \quad 0 \le n \le N.$$

(Higher derivatives) *If* $\mathcal{M}_n = \delta_{n0}$ *for* $0 \le n < M$ *then*

$$j^n\mathcal{M}_n = \frac{D^n}{n!}a(0) + j\frac{D^n}{n!}(p(\xi) - \gamma\xi)(0) \quad \textit{for} \quad 0 \le n < 2M. \tag{12}$$

The value of the delay γ affects the overall response of the filter. If $H(z) = \sum_{k=0}^{N} h_k z^k$ with $H(1) = 1$, the delay γ equals $\sum_{k=0}^{N} k h_k$. From [8, Proposition 3.2], we obtain

Proposition 4

$$\textit{If} \quad \max_{\xi \in \mathbf{R}} |H(e^{j\xi})| \leq 1 \quad \textit{then} \quad 0 \leq \gamma \leq N. \tag{13}$$

This result is not evident because the coefficients h_k are not necessarily positive and then γ does not need to be its center of mass. The reciprocal of (13) is not true as can be seen by choosing any symmetric polynomial with magnitude response not bounded by unity.

If the center of the passband is at a frequency $\varphi \in (0, \pi)$, a simple generalization of Theorem 1 leads to the following system for ALP around φ,

$$\sum_k h_k (k - \gamma)^{2n+1} e^{jk\varphi} = 0 \quad \text{for} \quad 0 \leq n < N.$$

For real coefficients $\{h_k\}$, we have the $2N$ equations,

$$\begin{cases} \sum_k h_k (k-\gamma)^{2n+1} \cos(k\varphi) = 0 \\ \sum_k h_k (k-\gamma)^{2n+1} \sin(k\varphi) = 0 \end{cases}$$

where $0 \leq n < N$.

IV Description of all approximate linear phase FIR digital filters

For real γ and nonnegative integers t and N let

$$\mathcal{L}_N^{\gamma,t} = \{P \in \mathbf{R}_N[X] : (xD)^{2n+1}(x^{-\gamma} P(x))(1) = 0 \quad \text{for} \quad 0 \leq n < t\}$$
$$\mathcal{L}_{\{N\}}^{\gamma,t} = \{P \in \mathcal{L}_N^{\gamma,t} : \deg(P) = N \quad \text{and} \quad P(1) = 1\}.$$

When $t = N$ we drop the superscript N as in $\mathcal{L}_{\{N\}}^{\gamma,N} = \mathcal{L}_{\{N\}}^{\gamma}$.

A filter H in $\mathcal{L}_{\{N\}}^{\gamma,t}$ has ALP of order t. We will see that the order t cannot be greater than N except if H has *exact* linear phase. In that case γ necessarily belongs to $[0 : N : \frac{1}{2}]$. For γ outside $[0 : N : \frac{1}{2}]$ there is only one polynomial in $\mathcal{L}_{\{N\}}^{\gamma}$.

Theorem 5 *Let γ a real number and N a nonnegative integer. Then*

$$\mathcal{L}_{\{N\}}^{\gamma} = \begin{cases} \emptyset & \textit{if} \quad \gamma \in [0 : \frac{N}{2} : \frac{1}{2}) \\ z^K \mathcal{R}_{\{N-K\}} & \textit{if} \quad \gamma = \frac{N+K}{2} \quad \textit{with} \quad 0 \leq K \leq N \\ \{L_N^{\gamma}\} & \textit{if} \quad \gamma \notin [0 : N : \frac{1}{2}] \end{cases}$$

where

$$L_N^\gamma(z) = \frac{1}{\binom{2N}{N}} \sum_{k=0}^{N} \binom{2\gamma}{k}\binom{2N-2\gamma}{N-k} z^k.$$

Furthermore, the set of polynomials of degree N and exact linear phase coincide with the set

$$\bigcup_{0\le K\le N} \mathcal{L}_{\{N\}}^{\frac{N+K}{2}}.$$

Proof See Appendix C.

In order to describe the set $\mathcal{L}_{\{N\}}^{\gamma,t}$, we first obtain all $A \in \mathcal{L}_N^{\gamma,t}$ and then ask for the normalization $A(1) = 1$. For most γ, $\mathcal{L}_N^{\gamma,t}$ turns out to be a subspace of $\mathcal{R}_N[X]$ of dimension $N+1-t$, but its description is different depending on whether γ lies inside or outside the set $[0 : N : \frac{1}{2}]$.

Corollary 6 *Let N a nonnegative integer and γ a real number outside $[0 : N : \frac{1}{2}]$. Then for $0 \le t \le N$, $\{L_k^\gamma\}_{k=t}^N$ is a basis of $\mathcal{L}_N^{\gamma,t}$ and for $t > N$, $\mathcal{L}_N^{\gamma,t} = \{0\}$.*

Proof Let $0 \le t \le N$. That the dimension of $\mathcal{L}_N^{\gamma,t}$ is $N+1-t$ can be obtained from the proof of Theorem 5. Since $\{L_k^\gamma\}_{k=t}^N$ are $N+1-t$ polynomials of different degrees in $\mathcal{L}_N^{\gamma,t}$, they are a basis for that space.

When $t > N$, assume $A \neq 0$ in $\mathcal{L}_N^{\gamma,t}$ and let $M \le N$ be the degree of A. Clearly $\mathcal{L}_M^{\gamma,t} \subseteq \mathcal{L}_t^\gamma$ and then $A(z) = \lambda L_t^\gamma(z)$ for some constant λ. But $\deg L_t^\gamma = t > M$ and thus $\lambda = 0$, a contradiction. It follows that $\mathcal{L}_N^{\gamma,t} = \{0\}$. ∎

We still need to consider the case when γ belongs to $[0 : N : \frac{1}{2}]$ or simply $\gamma \in [0 : \frac{N}{2} : \frac{1}{2}]$ because (30) with $\alpha = -1$ implies

$$A \in \mathcal{L}_N^{\gamma,t} \iff z^N A(z^{-1}) \in \mathcal{L}_N^{N-\gamma,t}. \tag{14}$$

In the next corollary, we show that when $t > N-1-\gamma$, $\mathcal{L}_N^{\gamma,t}$ equals $\mathcal{R}_{2\gamma}$. Observe that t can be arbitrarily large because this case corresponds to *exact* linear phase. When $t \le N-1-\gamma$, $\mathcal{L}_N^{\gamma,t}$ equals the direct sum of $\mathcal{R}_{2\gamma}$ and the subspaces $z^{\gamma+1}\mathcal{L}_{N-1-\gamma}^{-1,t}$ or $z^{\gamma+\frac{1}{2}}\mathcal{L}_{N-\frac{1}{2}-\gamma}^{-\frac{1}{2},t}$ depending on whether γ is an integer or a half integer.

Corollary 7 *Let $\gamma \in [0 : \frac{N}{2} : \frac{1}{2}]$. Then,*

$$\mathcal{L}_N^{\gamma,t} = \begin{cases} \mathcal{R}_{2\gamma} \oplus z^{\gamma+1}\mathcal{L}_{N-1-\gamma}^{-1,t} & \textit{if} \quad \gamma \in [0 : \frac{N}{2} : 1] \\ \mathcal{R}_{2\gamma} \oplus z^{\gamma+\frac{1}{2}}\mathcal{L}_{N-\frac{1}{2}-\gamma}^{-\frac{1}{2},t} & \textit{if} \quad \gamma \in [\frac{1}{2} : \frac{N}{2} : 1]. \end{cases}$$

The dimension of $\mathcal{L}_N^{\gamma,t}$ *is*

$$\dim \mathcal{L}_N^{\gamma,t} = \begin{cases} N+1-t & \textit{if} \quad t \le N-1-\gamma, \\ \\ \gamma+1 & \textit{if} \quad t > N-1-\gamma \quad \textit{and} \quad \gamma \in [0 : \frac{N}{2} : 1], \\ \\ \gamma+\frac{1}{2} & \textit{if} \quad t > N-1-\gamma \quad \textit{and} \quad \gamma \in [\frac{1}{2} : \frac{N}{2} : 1]. \end{cases}$$

Proof Let $\gamma \in [0 : \frac{N}{2} : \frac{1}{2}]$. Clearly $\mathcal{R}_{2\gamma} \subseteq \mathcal{L}_N^{\gamma,t}$. With the convention $\delta = \frac{1}{2}$ if γ is a half integer and $\delta = 1$ if γ is an integer, we have

$$z^{\gamma+\delta} \mathcal{L}_{N-\delta-\gamma}^{-\delta,t} \subseteq \mathcal{L}_N^{\gamma,t},$$

because $(xD)^{2n+1}(x^{-\gamma}(x^{\gamma+\delta}A))(1) = 0$ for all $0 \le n < t$ and all $A \in \mathcal{L}_{N-\delta-\gamma}^{-\delta,t}$.

Also, if $A \in \mathcal{R}_{2\gamma} \cap z^{\gamma+\delta}\mathcal{L}_{N-\delta-\gamma}^{-\delta,t}$,

$$A(z) = z^{2\gamma}A(z^{-1}) \quad \text{and} \quad A(z) = z^{\gamma+\delta}B(z)$$

for some polynomial B. Therefore $B(z^{-1}) = z^{2\delta}B(z)$ and then $B = 0$ and consequently $A = 0$.

We claim that for any $A \in \mathcal{L}_N^{\gamma,t}$ there exist $R \in \mathcal{R}_{2\gamma}$ and $P \in \mathcal{L}_{N-\delta-\gamma}^{-\delta,t}$ such that $A = R + P$. Given A we choose R to match the first $\gamma+\delta$ coefficients of A and define P of degree at most $N-\delta-\gamma$ such that $A - R = z^{\gamma+\delta}P$. Because of the conditions on A and R, $P \in \mathcal{L}_{N-\delta-\gamma}^{-\delta,t}$.

For the dimensions note that $\mathcal{L}_{N-\delta-\gamma}^{-\delta,t} = \{0\}$, if $t > N-\delta-\gamma$, and that $\dim \mathcal{R}_{2\gamma} = \gamma+\delta$. ■

V Examples

The following examples illustrate the previous descriptions. We will use some results on hypergeometric functions and Stirling numbers [5].

V.1 Simultaneous amplitude and phase approximation of an ideal response

We use Corollary 3 to construct filters with flat amplitude and flat group delay around zero.

Observe that for any function A, the following four conditions, valid for all k, $0 \le k < N$, are equivalent:

$$(xD)^n(x^{-\gamma}A(x))(1) = \delta_{n0}, \tag{15}$$

$$(xD)^n A(1) = \gamma^n, \tag{16}$$

$$\frac{D^n}{n!}A(1) = \binom{\gamma}{n}, \tag{17}$$

$$D^n(x^{-\gamma}A(x))(1) = \delta_{n0}. \tag{18}$$

With (17), the maximally flat solution can be obtained as

$$A_N^\gamma(z) = \sum_{k=0}^{N} \binom{\gamma}{k} (z-1)^k. \tag{19}$$

For other methods to derive these solutions see [7, Page 40] and references therein. The polynomials A_N^γ approximate the ideal fractional delay filter

$$z^\gamma = \sum_{k=0}^{\infty} \binom{\gamma}{k} (z-1)^k.$$

The frequency response of z^γ has "optimal" flat amplitude and flat group delay around zero.

To see which values of the delay lead to exact linear phase, use (15) to obtain

$$A_N^{N-\gamma}(z) = A_N^\gamma(z^{-1})z^N,$$

and then either $\gamma = \frac{N}{2}$ or γ belongs to $[0 : N : 1]$ and $A_N^\gamma(z) = z^\gamma$. The case $\gamma = \frac{N}{2}$ corresponds to a particular case of Herrmann's linear phase maximal flat amplitude filters [6].

The phase and amplitude response of A_N^γ are related to the moments of the coefficients by

$$\frac{D^{2n+1}}{(2n+1)!}(p(\xi) - \gamma\xi)(0) = (-1)^n \mathcal{M}_{2n+1}^{A_N^\gamma},$$
$$\frac{D^{2n}}{(2n)!}a(0) = (-1)^n \mathcal{M}_{2n}^{A_N^\gamma},$$

where $0 \le n < N$.

In conclusion, the first N derivatives at $\xi = 0$ of both $a(\xi)$ and $p(\xi) - \gamma\xi$ do vanish, and the next N derivatives can be computed in terms of higher moments of the coefficients of A_N^γ.

We now consider the problem of obtaining a FIR filter H, $H(e^{j\xi}) = a(\xi)e^{-jp(\xi)}$ with different flatness parameters N_p and N_a ($N_a \le 2N_p$):

$$D^{2n+1}p(0) = \gamma\delta_{n0} \quad \text{for} \quad 0 \le n < N_p, \quad \text{and}$$
$$D^{2n}a(0) = \delta_{n0} \quad \text{for} \quad 0 \le n < N_a.$$

According to Corollary 3, the problem is equivalent to

$$\mathcal{M}_{2n+1}^H = 0 \quad \text{for} \quad 0 \le n < N_p, \quad \text{and}$$
$$\mathcal{M}_{2n}^H = \delta_{n0} \quad \text{for} \quad 0 \le n < N_a.$$

Write

$$H(z) = \sum_{\min\{N_p,N_a\}}^{\max\{N_p,N_a\}} \lambda_k A_k^\gamma(z),$$

where λ_k are constants to be determined. The properties of A_N^γ yield,

$$\mathcal{M}_n^H = 0 \quad \text{for} \quad 0 < n < \min\{N_p, N_a\},$$

and we obtain $\mathcal{M}_0^H = 1$ by setting $\sum_k \lambda_k = 1$. The additional constrains on λ_k depend on higher moments of A_N^γ and can be expressed as,

If $N_a > N_p$

$$\mathcal{M}_{2n}^H = \sum_{k=N_p}^{N_a} \lambda_k \mathcal{M}_{2n}^{A_k^\gamma} \quad \text{for} \quad N_p \leq n < N_a.$$

If $N_p > N_a$

$$\mathcal{M}_{2n+1}^H = \sum_{k=N_a}^{N_p} \lambda_k \mathcal{M}_{2n+1}^{A_k^\gamma} \quad \text{for} \quad N_a \leq n < N_p.$$

To compute higher moments, we can use the expansion of A_N^γ around $z = 0$,

$$\begin{aligned} A_N^\gamma(z) &= \frac{\gamma^{N+1}}{N!} \sum_{k=0}^{N} \binom{N}{k} \frac{(-1)^{N+k}}{\gamma - k} z^k & (20) \\ &= \binom{\gamma - 1}{N} (-1)^N F(-\gamma, -N; 1-\gamma; z), & (21) \end{aligned}$$

where $F(a, b; c; z)$ is the hypergeometric series defined in the introduction.

Then, if L is a nonnegative integer,

$$(xD)^{N+1+L}(x^{-\gamma} A_N^\gamma(x))(1) = -\gamma^{N+1} \sum_{k=0}^{L} \binom{N+L}{k} S_k^{N+L-k} (-\gamma)^k, \qquad (22)$$

where S_k^n are the Stirling numbers of the second kind.

The LHS of Eq. (22) vanishes for all γ in $[0 : N : 1]$ because those choices of γ lead to exact linear phase and therefore all odd moments of A_N^γ should vanish. Different choices of γ will significantly impact on the values of higher moments of A_N^γ and consequently on the values of higher derivatives of its amplitude and phase.

In Figure 1 we plotted the frequency response characteristics for A_N^γ with $\gamma = 3.1$ and $N = 8$. Because of (13), the value of γ was chosen in the interval $[0, 8]$. For these filters both the amplitude and the group delay are flat around zero. For clarity, the values of the group delay in the passband have been shifted to zero.

V.2 FIR filters with maximally flat group delay

For each γ outside $[0 : N : \frac{1}{2}]$, the polynomials L_N^γ defined in Theorem 5 have maximally flat group delay within $\mathbf{R}_N[X]$. For γ in $[0 : N : \frac{1}{2}]$, there are infinitely many optimal ALP solutions, including symmetric polynomials leading to exact linear phase.

As pointed out in [12], the polynomials $L_N^\gamma(z)$ are related to hypergeometric and Legendre functions. In our notation,

$$L_N^\gamma(z) = \frac{\binom{2N-2\gamma}{N}}{\binom{2N}{N}} F(-2\gamma, -N; N+1-2\gamma; z). \tag{23}$$

Note the advantage of our formulation over the one by Thiran. By simply considering a different normalization, $L_N^\gamma(1) = 1$ as oppose to $L_N^\gamma(0) = 1$, $L_N^\gamma(z)$ becomes also a polynomial in the variable γ. To illustrate this advantage, we now show how to derive the value of L_N^γ at the Nyquist frequency. We claim

$$L_N^\gamma(-1) = 4^N \frac{N!}{(2N)!}(\frac{1}{2} - \gamma)(\frac{3}{2} - \gamma) \cdots (\frac{2N-1}{2} - \gamma).$$

First note that when γ belongs to $\{\frac{1}{2}, \frac{3}{2}, \cdots, \frac{2N-1}{2}\}$, $L_N^\gamma(z)$ is a symmetric polynomial of degree 2γ and then its value at -1 is zero. Thus,

$$L_N^\gamma(-1) = c_N(\frac{1}{2} - \gamma)(\frac{3}{2} - \gamma) \cdots (\frac{2N-1}{2} - \gamma),$$

for some constant c_N. Evaluating the previous equation at $\gamma = 0$,

$$1 = c_N \frac{4^{-N}}{N!} 2^N N! \, (1.3 \cdots (2N-1)),$$

we obtain the value of c_N.

Similarly to the case of simultaneous approximation, we can use linear combinations of the polynomials $L_N^\gamma(z)$ to generate filters with any order of ALP but satisfying additional properties.

In Figure 2 we plotted the frequency response characteristics for L_N^γ with $\gamma = 3.1$ and $N = 8$. The amplitude response is not flat around zero but its group delay is closer to constant when compared with the delay of A_N^γ.

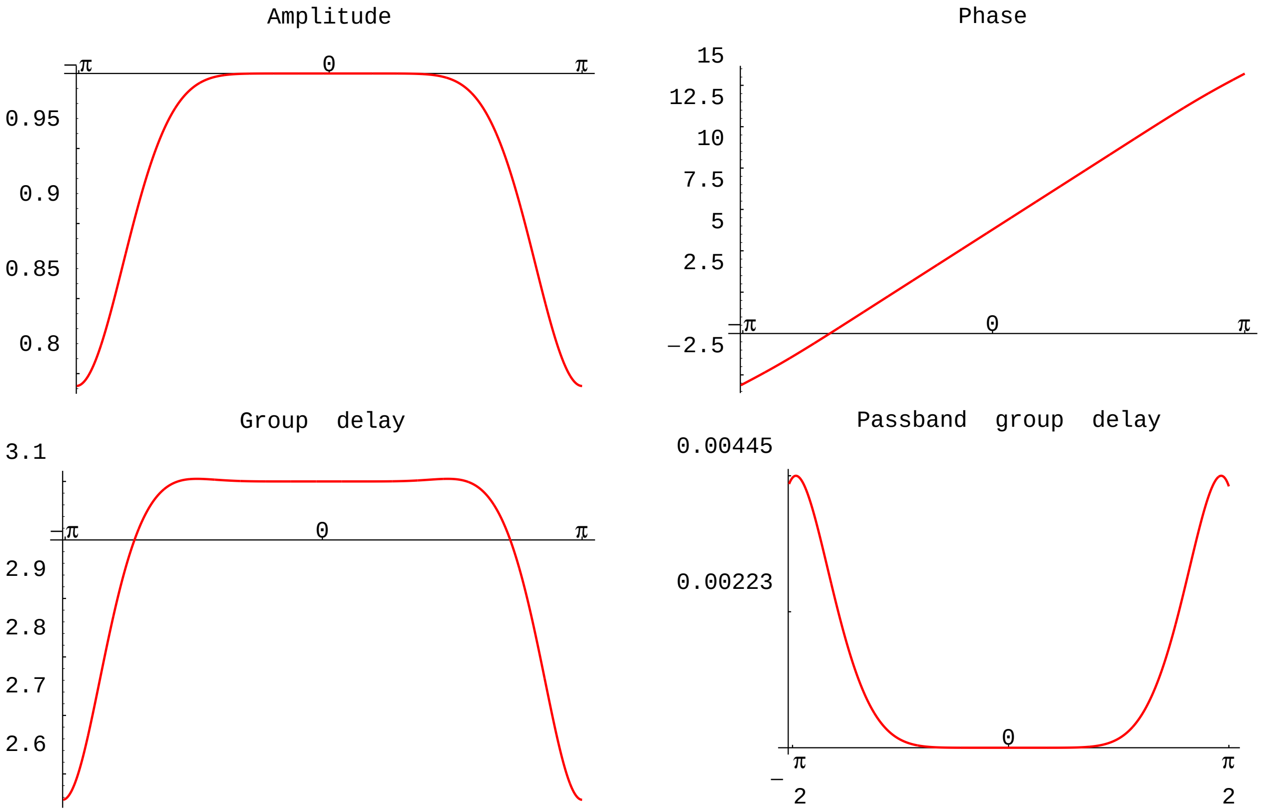

Figure 1: Frequency characteristics of optimal filter A_N^γ, with $\gamma = 3.1$ and $N = 8$.

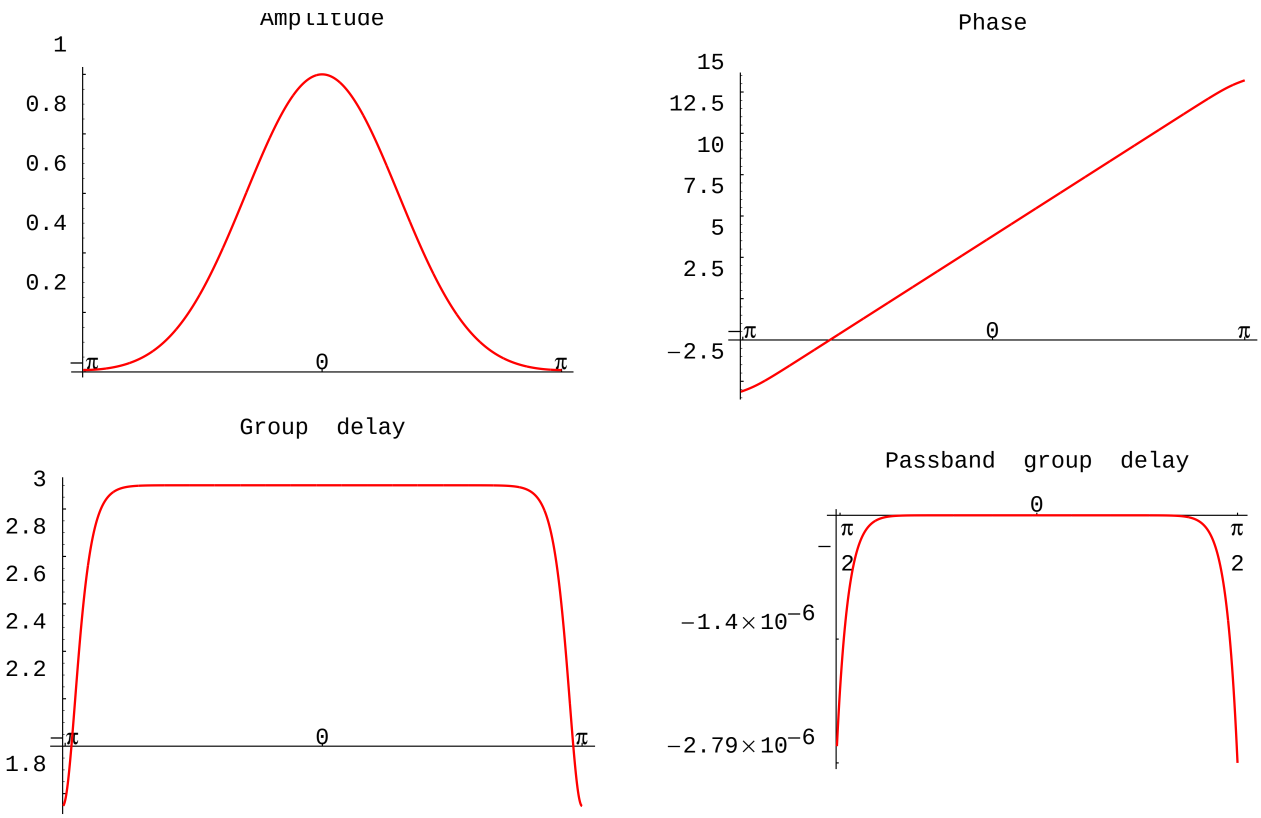

Figure 2: Frequency characteristics of maximally flat delay filter L_N^γ, with $\gamma = 3.1$ and $N = 8$.

VI Conclusion

A new approach to the theory and design of approximate linear phase filters has been presented. It is based on recognizing the existence of a linear formulation for phase approximation as well as for simultaneous amplitude and phase approximation.

This characterization provides linear conditions on the filter coefficients whether the low-pass filter is FIR, IIR (rational or non-rational), analog or digital. It also allows for arbitrary real values of the DC group delay.

In this paper we examined some consequences of this characterization for FIR digital filters. The examples presented intend to illustrate the advantages of this formulation and also provide the explicit building blocks for other approximate linear phase designs.

APPENDICES

A A result on functions with vanishing even derivatives

The derivative of a composition can be computed via

$$D^n(f \circ u)(z) = \sum_{k=0}^{n} P_k^n(z) D^k f(u(z)), \tag{24}$$

where

$$P_k^n(z) = \frac{n!}{k!} \sum_{\substack{j_i \geq 1 \\ j_1+\cdots+j_k=n}} \frac{D^{j_1}}{j_1!}u(z) \cdots \frac{D^{j_k}}{j_k!}u(z). \tag{25}$$

Lemma 8 *Assume α to be any real number and f and u to be functions such that $Df(u(\alpha)) \neq 0$. The following conditions are equivalent for all n, $0 \leq n < N$,*

$$D^{2n+1}u(\alpha) = 0, \quad \textit{and}$$
$$D^{2n+1}(f \circ u)(\alpha) = 0.$$

Proof Let $u_j = \frac{D^j}{j!}u(\alpha)$ and assume $u_1 = \cdots = u_{2N-1} = 0$. With (25), for $0 \leq n < N$, all terms in $P_k^{2n+1}(\alpha)$ contain and index i such that j_i is odd and $j_i \leq 2N-1$.

For the reciprocal, the case $N = 1$ is clear. If the statement is true for $n < N$, let $n = N$. Then

$$u_1 = \cdots = u_{2N-1} = 0 \quad \text{and}$$
$$0 = D^{2N+1}(f \circ u)(\alpha) = \sum_{k=1}^{2N+1} P_k^{2N+1}(u)(\alpha) D^k f(u(\alpha)).$$

In the sum the only non-zero term corresponds to $k = 1$. Hence $u_{2N+1} = 0$ because for all n, $P_1^n = D^n u$.

■

B Some properties of Vandermonde matrices

Let Γ be a set of n different complex numbers, $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$ and denote,

$$\Gamma^2 = \{\gamma_0^2, \gamma_1^2, \ldots, \gamma_{n-1}^2\} \quad \text{and} \quad a\Gamma + b = \{a\gamma_0 + b, a\gamma_1 + b, \ldots, a\gamma_{n-1} + b\},$$

for constants a and b. For $0 \le k < n$, let $C_{n,k}^{\Gamma}(x) = \sum_{j=0}^{n-1} c_{k,j}^{\Gamma} x^j$ be the unique polynomial of at most degree $n-1$ such that

$$C_{n,k}^{\Gamma}(\gamma_j) = \delta_{jk} \quad \text{for} \quad 0 \le j < n.$$

Let V^{Γ} be the n by n Vandermonde matrix of entries $V_{kj}^{\Gamma} = \gamma_j^k$. Since Γ consists of n different numbers, V^{Γ} has an inverse matrix $(V^{\Gamma})^{-1}$ whose rows are the coefficients of $C_{n,k}^{\Gamma}$,

$$(V^{\Gamma})_{kj}^{-1} = c_{k,j}^{\Gamma}. \tag{26}$$

It can be easily verified that for constants a and b,

$$C_{n,k}^{a\Gamma+b}(x) = C_{n,k}^{\Gamma}(\frac{x-b}{a}) \tag{27}$$

and, if Γ^2 has exactly n elements,

$$C_{n,k}^{\Gamma^2}(x^2) = \frac{C_{n,k}^{\Gamma}(x) C_{n,k}^{\Gamma}(-x)}{C_{n,k}^{\Gamma}(-\gamma_k)}. \tag{28}$$

For $\Gamma = \{0, 1, \cdots, n-1\}$ we simply write $C_{n,k}^{\Gamma} = C_{n,k}$. We have

$$C_{n,k}(x) = \binom{x}{k}\binom{n-1-x}{n-1-k} = \frac{(-1)^{n-1-k}}{(n-1)!}\binom{n-1}{k}\frac{x^{\underline{n}}}{x-k}. \tag{29}$$

C Proof of Theorem 5

For a function f and a real number α

$$(xD)^n f(x^{\alpha})(1) = \alpha^n (xD)^n f(1). \tag{30}$$

When $\alpha = -1$ and $f(x) = A(x) \in \mathcal{L}_{\{N\}}^{\gamma}$, we have

$$(xD)^k (x^{-\gamma} A(x) - x^{\gamma} A(x^{-1}))(1) = 0$$

for all $k, 0 \le k \le 2N$. Equivalently, $D^k(A(x) - x^{2\gamma} A(x^{-1}))(1) = 0$, for all $k, 0 \le k \le 2N$. When γ belongs to $[0 : N : \frac{1}{2}]$, $A(z) - z^{2\gamma} A(z^{-1})$ is a polynomial of degree at most $2N$. Thus

$$A(z) = z^{2\gamma} A(z^{-1}). \tag{31}$$

Since A has degree N, we have $0 \le 2\gamma - N \le N$ and then $\mathcal{L}^{\gamma}_{\{N\}} = \emptyset$ for all $\gamma \in [0 : \frac{N}{2} : \frac{1}{2})$. When $\gamma = \frac{K+N}{2}$ for $0 \le K \le N$, (31) implies $A(z) = z^K B_{N-K}(z)$ for some $B_{N-K} \in \mathcal{R}_{\{N-K\}}$. Reciprocally, a polynomial of degree N and exact linear phase satisfies (31) for integer 2γ and $0 \le 2\gamma - N \le N$. The last part of the theorem is proved.

Let γ be outside of $[0 : N : \frac{1}{2}]$ and $\{a_k\}_{k=0}^{N}$ be the coefficients of A in $\mathcal{L}^{\gamma}_{\{N\}}$. Eq. (5) is equivalent to the following Vandermonde system

$$\sum_{k=0}^{N-1} \gamma_k^{2n} b_k = \lambda^{2n}, \quad \text{for} \quad 0 \le n \le N-1 \tag{32}$$

where $b_k = \frac{k-\gamma}{\gamma-N}\frac{a_k}{a_N}$, $\gamma_k = k - \gamma$, and $\lambda = N - \gamma$.

Our assumption on γ yields $\gamma_k^2 \ne \gamma_{k'}^2$ if $k \ne k'$. Thus, there is only one solution $\{b_k\}$ of (32) that can be computed using Equations (26)-(28). If $\Gamma = \{\gamma_k\}$, for all $k,\ 0 \le k < N$,

$$\begin{aligned} b_k &= \sum_{r=0}^{N-1} c^{\Gamma^2}_{k,r}\lambda^{2r} = C^{\Gamma^2}_{N,k}(\lambda^2) = \frac{C^{\Gamma}_{N,k}(\lambda)C^{\Gamma}_{N,k}(-\lambda)}{C^{\Gamma}_{N,k}(-\gamma_k)} \\ &= \frac{C_{N,k}(\gamma+\lambda)C_{N,k}(\gamma-\lambda)}{C_{N,k}(2\gamma-k)} = \frac{C_{N,k}(N)C_{N,k}(2\gamma-N)}{C_{N,k}(2\gamma-k)}. \end{aligned}$$

To evaluate $C_{N,k}$ we use (29). It follows that $C_{N,k}(N) = \binom{N}{k}(-1)^{N-1-k}$ and

$$\begin{aligned} \frac{C_{N,k}(2\gamma-N)}{C_{N,k}(2\gamma-k)} &= \frac{(2\gamma-N)^{\underline{N}}(2\gamma-2k)}{(2\gamma-k)^{\underline{N}}(2\gamma-N-k)} \\ &= \frac{(2N-1-2\gamma)^{\underline{N}}}{(2\gamma-k)^{\underline{N+1}}}(-1)^N(2\gamma-2k) \end{aligned}$$

because $x^{\underline{n}} = (-1)^n(n-1-x)^{\underline{n}}$.

Writing back b_k in terms of a_k, for $0 \le k < N$,

$$a_k = a_N\binom{N}{k}(-1)^{k+1}\frac{(2N-2\gamma)^{\underline{N+1}}}{(2\gamma-k)^{\underline{N+1}}} = \frac{a_N}{\binom{2\gamma}{N}}\binom{2\gamma}{k}\binom{2N-2\gamma}{N-k}. \tag{33}$$

Note that (33) is also valid for $k = N$ and that

$$\sum_{k=0}^{N}\binom{x}{N-k}\binom{y}{k} = \binom{x+y}{N},$$

for all x, y [5, Eq. 5.22]. Thus, to obtain $\sum_{k=0}^{N} a_k = 1$ we choose $a_N = \binom{2\gamma}{N}/\binom{2N}{N}$. ■

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