Reduction of collision-induced timing shifts in dispersion-managed quasi-linear systems with periodic-group-delay dispersion compensation

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Periodic-group-delay (PGD) dispersion-compensation-modules have recently been proposed as mechanisms to alleviate collision-induced timing shifts in dispersion-managed (DM) systems. Frequency and timing shifts in quasi-linear DM systems with PGDs are obtained and it is shown that significant reductions are achieved when even a small fraction of the total dispersion is compensated by PGDs. (*Submitted to Optics Letters, May 6 2004.*)

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Collision-induced timing shifts – that is changes in the arrival time of a signal due to interaction with signals in neighboring frequency channels – is a major issue in on-off-keying, wavelength-division-multiplexed (WDM) systems. The problem is complex if dispersion management is employed: due to the periodic change with distance of the sign of chromatic dispersion, pulses in different frequency channels undergo a zig-zag motion with respect to each other, which results in interactions consisting of a series of "mini-collisions." Furthermore, the low value of average dispersion in these systems results in large collision lengths, implying the interaction unfolds over long distances. Indeed, while it is well-known that a moderate amount of dispersion management can be beneficial viz. interchannel interactions, in extreme cases the collision process can result in the mutual destruction of the two WDM pulses[1].

Recently, use of periodic-group-delay (PGD) modules has been proposed as a means of reducing collision-induced timing shifts[2–4]. PGD dispersion-compensating modules[5–7] are inserted inside each dispersion map, and serve to compensate for a fraction of the accumulated dispersion, with the remaining fraction of dispersion compensation still being achieved by dispersion compensating fibers (DCFs). The advantage of this approach for dispersion-managed soliton systems has been demonstrated both with numerical simulations[2,4] and with direct transmission experiments[3]. The purpose of this paper is twofold. First, we derive expressions for the value of the collision-induced frequency and timing shifts in quasi-linear return-to-zero systems which employ PGDs. Second, we use both our analytical results and direct numerical simulations to show that the use of PGDs is very effective in reducing the collision-induced timing shifts in quasi-linear, return-to-zero systems. As with solitons, the main mechanism for reduction in timing shift is the drastic reduction of the collision length[2]. We show that there also is a significant direct effect from the PGDs themselves, which provide a restoring force to the timing shift.

Pulse propagation in systems with PGDs. We start from the nonlinear Schrödinger (NLS) equation written in dimensionless variables, $t = t_{\mathrm{ret}}/t_*$, $z = z_{\mathrm{lab}}/z_*$, $u = \mathcal{E}\sqrt{g(z)P_*}$ and $D = -k''/k''_*$, with normalization parameters denoted by an asterisk. Here $\mathcal{E}$ is the slowly varying complex optical field envelope, t_{ret} and z_{lab} are, respectively, the retarded time and propagation distance. Typical normalization values used are $P_* = 1\,\mathrm{mW}$, $z_* = z_{\mathrm{NL}} = 1/(\gamma P_*) = 400\,\mathrm{km}$, $t_* = 12\,\mathrm{ps}$ and $k''_* = t_*^2/z_* = 0.36\,\mathrm{ps}^2/\mathrm{km}$, where $\gamma = 2.5\,(\mathrm{W{\cdot}km})^{-1}$ is the fiber nonlinear coefficient. We consider a dispersion map consisting of two fiber sections (with dispersion coefficients and lengths $D_1 > 0$, z_1 and $D_2 < 0$, z_2) and an Erbium-doped fiber amplifier (EDFA) located after the second fiber. The total map length is $z_a = z_1 + z_2$, and the fraction of the map consisting of the anomalous fiber is $\theta = z_1/z_a$. Here $g(z)$ describes the periodic power variation due to loss and amplification, and $D(z)$ is the local group velocity dispersion, both periodic functions with period z_a. With EDFAs, $g(z) = g_0 \exp(-2\Gamma z)$ for $nz_a < z < (n+1)z_a$, where n is the map number, Γ is the dimensionless loss coefficient and $g_0 = 2\Gamma z_a/(1-\exp(-2\Gamma z_a))$. A measure of the effects of dispersion management is given by the map strength[8] $s = [(D_1 - \langle D\rangle)z_1 - (D_2 - \langle D\rangle)z_2]/4$, where $\langle D\rangle = (D_1 z_1 + D_2 z_2)/z_a$ is the average dispersion. For a single pulse with normalized frequency Ω_0, its mean position $\langle t\rangle$ at distance z along the fiber is given by $\langle t\rangle = \Omega_0 \int_0^z D(x)dx = \Omega_0[\langle D\rangle z + C(z)]$, where $C(z) = \int_0^z [D(x) - \langle D\rangle]\,dx$ is periodic. Suppose two pulses $u_\pm$ with frequencies $\pm\Omega_0$ are initially located at $\mp t_0$; the initial time displacement t_0 is related to the mean collision location z_0 by $t_0 = \Omega_0\langle D\rangle z_0$. Mini-collisions between the pulses then occur when $(z - z_0)\langle D\rangle + C(z) = 0$. We can estimate the collision length to be $L_c = 2s/\langle D\rangle$. Complete collisions are those for which $L_c/2 \le z_0 \le L - L_c/2$.

In systems with PGDs, a PGD module is inserted in each dispersion map, providing a local dispersive force with strength $H_{\rm pgd}$ and replacing a fraction f of the dispersion provided by the second fiber. That is, $f = H_{\rm pgd}/(D_2 z_2' + H_{\rm pgd})$, where hereafter a prime denotes a quantity in a system with PGDs. We consider this replacement to be done while keeping the average dispersion, the individual dispersions and the length of the first fiber section to be unchanged: $D_{1,2}' = D_{1,2}$, $z_1' = z_1$, and $\langle D\rangle' = (D_1 z_l' + D_2 z_2' + H_{\rm pgd})/z_a' = \langle D\rangle$. We then find the new system parameters to be $\theta' = [(D_1 - D_2)(1-f)\theta - D_2 f]\,/\,[D_1(1-f) - D_2]$, $z_a' = (\theta/\theta')z_a$, and $z_2' = (1-\theta')z_a'$. It is useful to define an effective average dispersion by $\langle D\rangle_{\rm eff} \equiv (D_1 z_1 + D_2 z_2')/z_a'$. Using the equivalent definition of map strength $2s = (D_1 - \langle D\rangle)\,z_1$, we then find the new map strength given by $s' = [(D_1 - \langle D\rangle_{\rm eff})/(D_1 - \langle D\rangle)]\,s$. The new collision length is then $L_c' = 2s'/\langle D\rangle_{\rm eff}$. Propagation of optical pulses in such a system is governed by a perturbed NLS equation:

$$iu_z + \tfrac{1}{2}D(z)u_{tt} + g(z)|u|^2 u = iP[u]\,, \tag{1a}$$

$$\hat{P}[u] = (e^{iH(\omega)} - 1)\sum_{m=1}^{N_a} \delta(z - mz_a')\hat{u}(\omega, z) \tag{1b}$$

(cf. Ref. 9), where $P[u]$ expresses the action of the PGDs in the time domain, $\hat{P}[u]$ is the Fourier transform of $P[u]$ and $\hat{u}(\omega,z)$ that of $u(t,z)$, and where N_a is the total number of dispersion maps in the transmission line. The function $H(\omega)$ is the PGD response, which locally approximates a quadratic dispersive profile but is periodic with period equal to the channel spacing, in our case $2\Omega_0$. Finally, $\delta(z)$ is the Dirac delta, which encodes the pulse change across a PGD:

$\hat{u}(\omega, mz_a'^{+}) = e^{iH(\omega)}\hat{u}(\omega, mz_a'^{-})$.

Collision-induced frequency and timing shifts with PGDs. Inserting $u = u_+ + u_-$ in Eqs. (1) and neglecting four-wave mixing, one finds the evolution for $u_\pm$ as Eq. (1) with an extra term on the left-hand-side: namely, $2g(z)|u_\mp|^2u_\pm$ in the equation for $u_\pm$. The mean time and pulse frequency are defined as usual by $\langle t\rangle = \int t|u_\pm|^2dt/E$ and $\Omega = -i\int u_\pm^*(\partial u_\pm/\partial t)dt/E$, where $E = \int |u_\pm|^2dt$ is the pulse energy. (All integrals are from $-\infty$ to ∞ unless otherwise indicated.) Using Eqs. (1) and a second-order Taylor approximation of $\hat{P}[u]$ about $\omega = \Omega_0$ with $H(\Omega_0) = H'(\Omega_0) = 0$, we find evolution equations for the mean time and frequency. Looking at u_+:

$$\frac{\partial\Omega}{\partial z} = \frac{2g(z)}{E}\int |u_+|^2\frac{\partial|u_-|^2}{\partial t}dt\,, \tag{2a}$$

$$\frac{\partial\langle t\rangle}{\partial z} = D(z)\Omega(z) + H_{\rm pgd}\sum_{m=1}^{N_a}\delta(z - mz_a')\,(\Omega - \Omega_0)\,, \tag{2b}$$

where $H_{\rm pgd} = H''(\Omega_0)$. The timing and frequency shifts are then $\Delta\Omega = \Omega(z) - \Omega_0$ (with $\Omega(0) = \Omega_0$), and $\Delta t(z) = \langle t(z)\rangle - \tilde{D}(z)\Omega_0 + t_0$, where $\tilde{D}(z) \equiv \int_0^z D(x)dx$ would be the accumulated dispersion in the new system without PGDs. It is also convenient to define the dispersion including PGDs as $D_{\rm pgd}(z) = D(z) + H_{\rm pgd}\sum_{m=1}^{N_a}\delta(z - mz_a')$ and the accumulated dispersion including PGDs as $\tilde{D}_{\rm pgd}(z) = \int_0^z D_{\rm pgd}(x)dx$. The evolution with distance of a quasi-linear pulse with an initial Gaussian profile $u_\pm(t,0) = (a/\sqrt{2\pi b})\exp[-(t\pm t_0)^2/2b \pm i\Omega_0 t]$ is then given by[10]

$$u_\pm(t,z) = \frac{a}{\sqrt{2\pi(b + i\tilde{D}_{\rm pgd}(z))}}\exp\left[-\frac{(t\pm t_0 \mp \Omega_0\tilde{D}(z))^2}{2(b + i\tilde{D}_{\rm pgd}(z))} \pm i\Omega_0 t - \frac{i}{2}\Omega_0^2\tilde{D}(z)\right]. \tag{3}$$

Using Eq. (3) in Eqs. (2) we find

$$\Delta\Omega(L) = A\Omega_0\int_0^L \frac{g(z)\tilde{D}_0(z)}{\left(b^2 + \tilde{D}_{\rm pgd}^2(z)\right)^{3/2}}\exp\left(-2b\Omega_0^2\frac{\tilde{D}_0^2(z)}{b^2 + \tilde{D}_{\rm pgd}^2(z)}\right)dz \tag{4a}$$

and

$$\Delta t(L) = \tilde{D}_0(L)\Delta\Omega(L) - \Delta t_{\rm res} + H_{\rm pgd}\sum_{m=1}^{N_a}\Delta\Omega(mz_a'), \tag{4b}$$

where $\tilde{D}_0(z) = \tilde{D}(z) - \langle D\rangle_{\rm eff}z_0$, with $A = 4Eb^{3/2}/\sqrt{2\pi}$ and where the "residual" timing shift $\Delta t_{\rm res}$ is

$$\Delta t_{\rm res}(L) = A\Omega_0\int_0^L \frac{g(z)\tilde{D}_0^2(z)}{\left(b^2 + \tilde{D}_{\rm pgd}^2(z)\right)^{3/2}}\exp\left(-2b\Omega_0^2\frac{\tilde{D}_0^2(z)}{b^2 + \tilde{D}_{\rm pgd}^2(z)}\right)dz\,. \tag{4c}$$

Equation (4b) shows there is an additional restoring force due to the PGDs. The sum in Eq. (4b) is calculated using Eq. (4a) with L replaced by mz_a'. Note that the above theory allows calculation of frequency and timing shifts for complete as well as incomplete collisions. Note also that

Eq. (2b) can be written in compact form as $\partial\langle t\rangle/\partial z = D_{\text{pgd}}(z)\Delta\Omega(z) + D(z)\Omega_0$. Importantly, the integrals in Eqs. (4a) and (4c) can be efficiently evaluated by numerical quadrature or by asymptotic approximation, using the so-called Laplace method (see Refs. 11,12).

Asymptotic approximation via the Laplace method. The integrals in Eqs. (4a) and (4c) have the generic form $I(\lambda) = \int_0^L F(z)e^{-\lambda\Phi(z)}dz$, where $\lambda > 0$ is a parameter. For large values of λ, the main contribution to $I(\lambda)$ comes from a neighborhood of a so-called critical point z_c where $\Phi(z)$ is minimum, other regions contributing exponentially smaller terms. The functions $F(z)$ and $\Phi(z)$ can then be expanded in Taylor series about the critical point, with typically only the first few terms retained.

In our case, $\Phi(z) = \tilde{D}_0^2/(b^2 + \tilde{D}_{\text{pgd}}^2)$ and $\lambda = 2b\Omega_0^2$ for both Eqs. (4a) and (4c). The critical points z_c correspond to mini-collision locations, which occur where $\tilde{D}_0(z_c) = 0$. For Eq. (4a), $F(z) = g\tilde{D}_0/(b^2 + \tilde{D}_{\text{pgd}}^2)^{3/2}$, while for Eq. (4c) $F(z) = g\tilde{D}_0^2/(b^2 + \tilde{D}_{\text{pgd}}^2)^{3/2}$. We thus have $F(z_c) = \Phi(z_c) = 0$ for both $\Delta\Omega$ and Δt_{res}, but $F'(z_c) \neq 0$ for $\Delta\Omega$, while $F'(z_c) = 0$ for Δt_{res}. Then, if $z_1, \ldots, z_N$ are the locations of the mini-collisions, we find

$$\Delta\Omega(L) \simeq A\sum_{n=1}^{N}\left[\frac{F_n'J_{1,n}}{b\Omega_0\Phi_n''} + \frac{3F_n''\Phi_n''J_{2,n} - 2F_n'\Phi_n'''J_{4,n}}{6b^{3/2}\Omega_0^2(\Phi_n'')^{5/2}}\right], \tag{5a}$$

$$\Delta t_{\text{res}}(L) \simeq \frac{A}{b^{3/2}\Omega_0^2}\sum_{n=1}^{N}\frac{F_n''J_{2,n}}{2(\Phi_n'')^{3/2}}, \tag{5b}$$

where $F_n' = F'(z_n)$ etc. and $J_{k,n} = \int_{x_n^-}^{x_n^+} x^k e^{-x^2}dx$, with the limits of integration $x_n^{\pm}$ depending on the location of the mini-collision. The integrals $J_{n,k}$ may be evaluated exactly in terms of well-known functions. Eqs. (5) show the residual frequency shift and the residual timing shift are, respectively, $O(1/\Omega_0)$ and $O(1/\Omega_0^2)$. The accuracy of the Laplace method can be further improved by keeping higher order terms.

Numerical simulations and discussion. We now compare the results of direct numerical simulations (DNS) of the NLS Eq. (1), the results of numerically integrating Eqs. (4) (quadrature) and asymptotically approximating them with Eqs. (5) (Laplace). Implicit in our quasi-linear ansatz is that the PGD approximates a quadratic dispersive profile. Thus, we used a piecewise, periodic function to model PGDs in Eqs. (1): $H(\omega,z) = (\omega - \Omega_0)^2 H_{\text{pgd}}/2$ for $2n\Omega_0 \leq \omega < 2(n+1)\Omega_0$. Since a PGD is assumed to act locally in space, the pulse was propagated linearly through the spatial cells containing the PGDs, with the jump condition given after Eqs. (1). We thus expect Eqs. (4) to be good approximations when the spectral width of the pulse is small compared to the

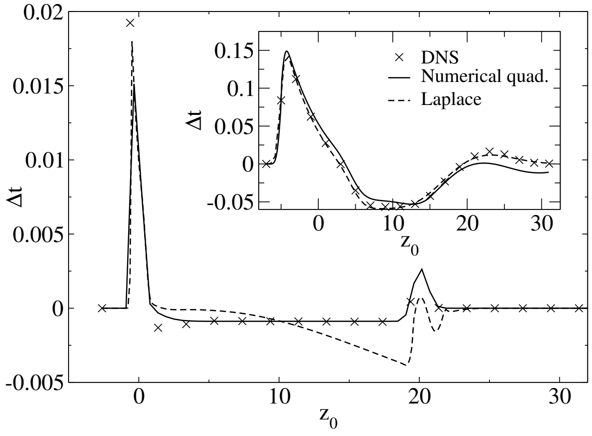

Fig. 1. Total timing shift versus the mean collision location z_0 for $f = 0.1$, obtained from direct numerical simulations (DNS) of Eq. (1), by numerically integrating (quadrature) Eqs. (4) or by approximating them (Laplace) via Eqs. (5). The inset shows a system without PGDs, $f = 0$.

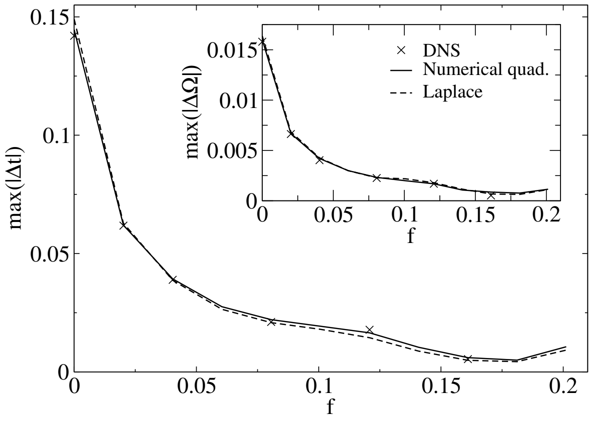

Fig. 2. Maximum timing shift and frequency shift (inset) with respect to the mean collision location z_0, as a function of f.

period of the PGDs, $2\Omega_0$. In the numerical quadrature, Eqs. (4) were evaluated by breaking the range of integration $[0,L]$ into subintervals of constant dispersion and using the trapezoidal rule on each subinterval.

We simulated a quasi-linear system with normalized line parameters $L = 20$, $\langle D \rangle = 0.5$, $s = 2.5$, $\theta = 0.5$, $\Gamma = 9.21$ and $z_a = 0.15$, and normalized pulse parameters $a = 1.6$, $b = 2$ and $\Omega_0 = 3$, These values correspond to a system with channel spacing 80 GHz, average dispersion 0.18 ps^2/km, total transmission distance 8,000 km, loss coefficient 0.2 dB/km, amplifier spacing 40 km without

PGDs, and an input pulse with a peak power of 0.6 mW and a full-width at half-maximum of 28.26 ps. Figure 1 shows the total timing shift $\Delta t(L)$ at the output versus the mean collision location z_0 for a system with ($f = 0.1$) and without ($f = 0$) PGDs. Figure 2 shows the maximum absolute value of the total timing shift and the residual frequency shift as a function of f, the fraction of compensation performed by the PGDs; the maximum is taken over the mean collision location z_0. In addition to the excellent agreement between analytical formulae and direct numerics, these figures clearly show the large reduction in the collision induced timing shifts arising from use of PGDs, even with values of f as small as 0.05. Note that the reduction of the timing shift corresponds to a sharp decrease in the collision length: in Fig. 1, $L_c' = 1.24$ for $f = 0.1$, compared to $L_c = 10$ for $f = 0$ (no PGDs). Note also that $L_c' \ll L$ when f is not too small, which means that, with PGDs, almost all collisions are complete, unlike the case without PGDs. Finally, Fig. 1 shows that, with PGDs, the effect of complete collisions is almost negligible (unlike the case without PGDs), and the only noticeable timing shifts arise from the few incomplete collisions, corresponding to collision centers z_0 located near the beginning ($z_0 = 0$) or the end ($z_0 = 20$) of the transmission line.

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