APPM 1340 **Final Exam Solutions** **Fall 2009**

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1. (a) -1 (b) UND (c) 0 (d) 2 (e) UND (f) $x = 2, 4$
   (g) $x = 0, 2, 4, 5$.

2. (a) $(f \circ g)(x) = f(g(x)) = f(x+2) = 1 + \sqrt{3(x+2) - 6} =$
   $1 + \sqrt{3x + 6 - 6} = \boxed{1 + \sqrt{3x}}$.

   (b)
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
(f\circ g)'(x) &= \lim_{h\to 0} \frac{1 + \sqrt{3(x+h)} - (1 + \sqrt{3x})}{h} \\
&= \lim_{h\to 0} \frac{1 + \sqrt{3x + 3h} - 1 - \sqrt{3x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3x+3h} - \sqrt{3x}}{h} \cdot \frac{\sqrt{3x+3h} + \sqrt{3x}}{\sqrt{3x+3h} + \sqrt{3x}} \\
&= \lim_{h\to 0} \frac{3x + 3h - 3x}{h(\sqrt{3x+3h} + \sqrt{3x})} \\
&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3x+3h} + \sqrt{3x})} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{3x+3h} + \sqrt{3x}} \\
&= \frac{3}{\sqrt{3x} + \sqrt{3x}} = \boxed{\frac{3}{2\sqrt{3x}}}.
\end{aligned}$$

3. (a) 5 (b) 3 (c) 2 (d) 4 (e) 1

4. (a) 2 (b) 4 (c) 5 (d) 1 (e) 3

5. (a) Counterexample: $f(x) = -|x+2|$.
   (b) True.
   (c) Counterexample: $f(x) = -x^3$.

6. (a) $\displaystyle \lim_{x\to 7^+} \frac{7-x}{x^2-49} = \lim_{x\to 7^+} \frac{7-x}{(x+7)(x-7)} = \lim_{x\to 7^+} \frac{-1}{x+7} = \boxed{-\frac{1}{14}}$.

   (b) $\displaystyle \lim_{t\to 5^-} \frac{t^2 - 3t - 4}{t-5} = \lim_{t\to 5^-} \frac{(t-4)(t+1)}{t-5} = \boxed{-\infty}$.

   (The expression is (pos · pos) / neg.)

   (c) $\displaystyle \lim_{\theta\to 0} 2\theta \sec 2\theta \tan 3\theta = \lim_{\theta\to 0} 2\theta \cdot \frac{1}{\cos 2\theta} \cdot \frac{\sin 3\theta}{\cos 3\theta} = 0 \cdot \frac{1}{1} \cdot \frac{0}{1} = \boxed{0}$.

7.
$$\begin{aligned}
(x^2+y^2)^2 &= 2(x^2-y^2) \\
x^4 + 2x^2y^2 + y^4 &= 2x^2 - 2y^2 \\
4x^3 + (2x^2)\left(2y\tfrac{dy}{dx}\right) + (y^2)(4x) + 4y^3\tfrac{dy}{dx} &= 4x - 4y\tfrac{dy}{dx} \\
4x^3 + 4x^2y\tfrac{dy}{dx} + 4xy^2 + 4y^3\tfrac{dy}{dx} &= 4x - 4y\tfrac{dy}{dx} \\
x^3 + x^2y\tfrac{dy}{dx} + xy^2 + y^3\tfrac{dy}{dx} &= x - y\tfrac{dy}{dx} \\
x^2y\tfrac{dy}{dx} + y^3\tfrac{dy}{dx} + y\tfrac{dy}{dx} &= x - x^3 - xy^2 \\
\tfrac{dy}{dx}\left(x^2y + y^3 + y\right) &= x - x^3 - xy^2 \\
\frac{dy}{dx} &= \boxed{\frac{x - x^3 - xy^2}{x^2y + y^3 + y}}.
\end{aligned}$$

8. (a) For the horizontal asymptote, we check the limit as x approaches $\pm\infty$.
   $\displaystyle \lim_{x\to\pm\infty} \frac{x+x^2}{1-4x^2} = \lim_{x\to\pm\infty} \frac{\frac{1}{x}+1}{\frac{1}{x^2}-4} = -\frac{1}{4}$.

   There is a horizontal asymptote at $\boxed{y = -\frac{1}{4}}$.

   For the vertical asymptotes, we check for zero denominator by solving
   $1 - 4x^2 = 0 \implies 4x^2 = 1 \implies x^2 = 1/4 \implies x = \pm\frac{1}{2}$.

   There are vertical asymptotes at $\boxed{x = \pm\frac{1}{2}}$.

(b) $y = \dfrac{x + x^2}{1 - 4x^2}$

$$y' = \frac{(1-4x^2)(1+2x) - (x+x^2)(-8x)}{(1-4x^2)^2}$$

The tangent slope at $x = -1$ is

$$y'|_{x=-1} = \frac{(1-4(-1)^2)(1+2(-1)) - (-1+(-1)^2)(-8(-1))}{(1-4(-1)^2)^2}$$
$$= \frac{(1-4)(-1) - (0)(8)}{(1-4)^2} = \frac{3-0}{9} = \frac{1}{3}.$$

At $x = -1$, $y = 0$. An equation for the tangent line is

$$y = y_1 + m(x - x_1)$$
$$y = 0 + \tfrac{1}{3}(x - (-1))$$
$$y = \tfrac{1}{3}x + \tfrac{1}{3}$$

9. $y = \tan x - x$

$$y' = \sec^2 x - 1 = \frac{1}{\cos^2 x} - 1$$
$$y'' = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$$
$$= 2 \cdot \frac{1}{\cos^2 x} \cdot \frac{\sin x}{\cos x}$$
$$= \frac{2\sin x}{\cos^3 x}$$

The first derivative y' equals 0 when $\cos^2 x = 1$ at $x = 0$. It is defined for all values of x in this interval.

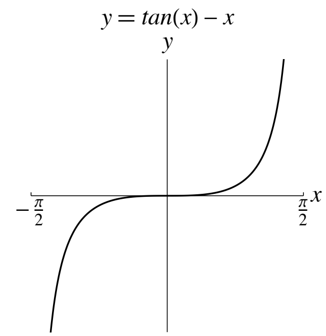
The second derivative y'' equals 0 when $\sin x = 0$ at $x = 0$. It is defined for all values of x in this interval.

Note that in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $0 < \cos x < 1$. In the interval $\left(-\frac{\pi}{2}, 0\right)$, $\sin x < 0$ and in the interval $\left(0, \frac{\pi}{2}\right)$, $\sin x > 0$.

y': + + + | + + +
$-\frac{\pi}{2}$ ↗ 0 ↗ $\frac{\pi}{2}$

y'': − − − | + + + +
$-\frac{\pi}{2}$ ∩ 0 ∪ $\frac{\pi}{2}$

The function is rising throughout the open interval. There is one inflection point at $(0, 0)$ and no local extrema.



10. The function g is continuous at $x = 0$ if $\lim_{x \to 0} g(x) = g(0) = 1$. We use the Sandwich Theorem to evaluate $\lim_{x \to 0} g(x)$.

$$-1 \le \cos\left(\tfrac{1}{x^3}\right) \le 1$$
$$-x^2 \le x^2 \cos\left(\tfrac{1}{x^3}\right) \le x^2$$

Since $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$, then $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^3}\right) = 0$.

Therefore g is *not* continuous at $x = 0$ because $\lim_{x \to 0} g(x) \ne g(0)$.

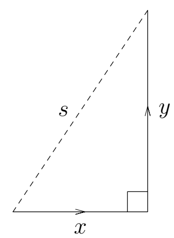
11. (a) $s(t) = 240t - 16t^2$

$$v(t) = 240 - 32t$$
$$v(0) = \boxed{240\,\text{ft/sec}}.$$

(b) $s(5) = 240(5) - 16\left(5^2\right) = 1200 - 400 = \boxed{800\,\text{ft}}$.

(c) $v(5) = 240 - 32(5) = 240 - 160 = \boxed{80\,\text{ft/sec}}$.

(d) Let x represent Lois' distance from the building, y represent Superman's height above the ground, and s represent the distance between Lois and Superman.



At this moment, since $x = 600$ and $y = 800$ (from 11b), then $s = 1000$. We also know that $dx/dt = -75$ and and $dy/dt = 80$ (from 11c). Then

$$\begin{aligned} s^2 &= x^2 + y^2 \\ 2s\frac{ds}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ s\frac{ds}{dt} &= x\frac{dx}{dt} + y\frac{dy}{dt} \\ 1000\frac{ds}{dt} &= (600)(-75) + (800)(80) \\ 10\frac{ds}{dt} &= (6)(-75) + (8)(80) \\ \frac{ds}{dt} &= \frac{-450 + 640}{10} = 19. \end{aligned}$$

The distance between Lois and Superman is *increasing* at the rate of $\boxed{\text{19 ft/sec}}$.

(e) The maximum height occurs when $v(t) = 0$.

$$\begin{aligned} 240 - 32t &= 0 \\ 32t &= 240 \\ t &= \tfrac{240}{32} = \tfrac{15}{2}. \end{aligned}$$

When $t = \frac{15}{2}$ sec, the maximum height is

$$\begin{aligned} s\left(\tfrac{15}{2}\right) &= 240\left(\tfrac{15}{2}\right) - 16\left(\tfrac{15}{2}\right)^2 \\ &= 120(15) - 4(15)(15) = 15(120 - 60) = 15(60) \\ &= \boxed{\text{900 ft}}. \end{aligned}$$

(f) Since Superman's maximum height is also the height of the Daily Planet building, he will reach the rooftop $\frac{15}{2} = 7.5$ sec after takeoff, or 2.5 sec after Lois is within 600 ft of the building. She will need $\frac{600}{75} = 8$ more seconds to reach the building.

Therefore Lois will *not* get to the front entrance before Superman reaches the rooftop.