

1. (a) The statement $\lim_{x \rightarrow x_0} f(x) = L$ is true if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

- (b) $x_0 = \frac{3}{2}, f(x) = \frac{1}{2}(x - 3), L = -\frac{3}{4}, \epsilon = \frac{1}{8}.$

$$\begin{aligned} |f(x) - L| < \epsilon \\ \left| \frac{1}{2}(x - 3) + \frac{3}{4} \right| < \frac{1}{8} \\ -\frac{1}{8} < \frac{1}{2}(x - 3) + \frac{3}{4} < \frac{1}{8} \\ -\frac{7}{8} < \frac{1}{2}(x - 3) < -\frac{5}{8} \\ -\frac{7}{4} < x - 3 < -\frac{5}{4} \\ \frac{5}{4} < x < \frac{7}{4} \end{aligned}$$

Now center the interval about $x_0 = \frac{3}{2}$. The distance from x_0 to either endpoint is $\frac{1}{4}$ so choose $\delta = \frac{1}{4}$. Then

$$0 < \left| x - \frac{3}{2} \right| < \frac{1}{4} \implies \left| \frac{1}{2}(x - 3) + \frac{3}{4} \right| < \frac{1}{8}.$$

2. (a) $\lim_{x \rightarrow -5^-} \frac{x^2 - 4x - 5}{x^2 + 4x - 5} = \lim_{x \rightarrow -5^-} \frac{(x - 5)(x + 1)}{(x + 5)(x - 1)} = \boxed{\infty}.$

(The expression is (neg · neg) / (neg · neg) = pos.)

- (b) $\lim_{\theta \rightarrow 2\pi^-} -\frac{2 \csc \theta}{\sec \theta} = \lim_{\theta \rightarrow 2\pi^-} \frac{-2/\sin \theta}{1/\cos \theta} = \lim_{\theta \rightarrow 2\pi^-} \frac{-2 \cos \theta}{\sin \theta} = \boxed{\infty}.$

(The expression is (neg · pos) / neg = pos.)

(Alternatively $\lim_{\theta \rightarrow 2\pi^-} -2 \cot \theta = \infty$.)

Note that when θ approaches 2π from the left side, $\sin \theta$ has negative values, and $\lim_{\theta \rightarrow 2\pi^-} \cos \theta = 1$, which is positive.

3. (a) Let $y = \frac{3}{1 - x}$. Then

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{3}{1 - (x+h)} - \frac{3}{1 - x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{3}{1 - x - h} - \frac{3}{1 - x} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{3(1 - x)}{(1 - x - h)(1 - x)} - \frac{3(1 - x - h)}{(1 - x)(1 - x - h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{3 - 3x - (3 - 3x - 3h)}{(1 - x)(1 - x - h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{3 - 3x - 3 + 3x + 3h}{(1 - x)(1 - x - h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{3h}{(1 - x)(1 - x - h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{3}{(1 - x)(1 - x - h)} \\ &= \boxed{\frac{3}{(1 - x)^2}}. \end{aligned}$$

- (b) Quotient Rule: $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}.$

For $y = \frac{3}{1 - x}$, $u = 3$ and $v = 1 - x$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 - x)(3)' - (3)(1 - x)'}{(1 - x)^2} \\ &= \frac{(1 - x)(0) - (3)(-1)}{(1 - x)^2} \\ &= \boxed{\frac{3}{(1 - x)^2}}. \end{aligned}$$

- (c) The curve has no horizontal tangents because $\frac{3}{(1 - x)^2} = 0$ has no solutions. The derivative does not equal 0 for any value of x .

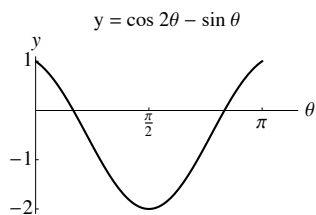
4. The Intermediate Value Theorem for continuous functions states that if function f is continuous on a closed interval $[a, b]$, then f takes on every value between $f(a)$ and $f(b)$.

The function $f(\theta) = \cos 2\theta - \sin \theta$ is a continuous function because $\cos 2\theta$ and $\sin \theta$ are both continuous for all θ .

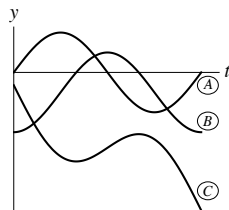
Let $f(\theta) = \cos 2\theta - \sin \theta$. Then

$$\begin{aligned} f(0) &= \cos 0 - \sin 0 = 1 - 0 = 1, \\ f\left(\frac{\pi}{2}\right) &= \cos \pi - \sin \frac{\pi}{2} = -1 - 1 = -2, \\ f(\pi) &= \cos 2\pi - \sin \pi = 1 - 0 = 1. \end{aligned}$$

By the IVT, since f takes on every value from 1 to -2 for $0 \leq \theta \leq \frac{\pi}{2}$, then $f(\theta) = 0$ for at least one value of θ in $[0, \frac{\pi}{2}]$. Note that there is a second solution in $[\frac{\pi}{2}, \pi]$.



5.



We examine the *slope* of each function to determine the *values* of its derivative.

The graph of C has neg-pos-neg slope, which corresponds to the neg-pos-neg values of B , so B is the derivative of C .

The graph of B has pos-neg slope, which corresponds to the pos-neg values of A , so A is the derivative of B .

The graph of A has pos-neg-pos slope, which does not correspond to the values of B or C .

We conclude that C is the position function, B is the velocity function, and A is the acceleration function.

6. (a)

$$f(x) = \begin{cases} ax^2 - 6, & x \leq 4 \\ -4\sqrt{x}, & x > 4. \end{cases}$$

For the LH function, $f(4) = a(4^2) - 6 = 16a - 6$.

For the RH function, $\lim_{x \rightarrow 4} -4\sqrt{x} = -4\sqrt{4} = -8$.

Function f is continuous if the LH and RH values are equal:

$$16a - 6 = -8 \implies 16a = -2 \implies \boxed{a = -\frac{1}{8}}.$$

(b) Function f is differentiable at $x = 4$ if the LH and RH derivatives are equal.

LH derivative: $m_1 = 2ax = 2\left(-\frac{1}{8}\right)(4) = -1$.

RH derivative: $m_2 = -\frac{2}{\sqrt{x}} = -\frac{2}{\sqrt{4}} = -\frac{2}{2} = -1$.

Since the LH and RH derivatives both equal -1 , f is differentiable at $x = 4$.

7. (a) Moving to the right: $(2, 3)$, $(5, 6)$

(b) Slowing down: $(1, 2)$, $(2\frac{1}{2}, 3)$, $(4, 4\frac{1}{2})$, $(5\frac{1}{2}, 6)$

(c) Positive acceleration: $(1, 2\frac{1}{2})$, $(4, 4\frac{1}{2})$, $(5, 5\frac{1}{2})$

(d) Maximum speed: $t = 4$

8. (a)

$$s(t) = 81 - 16t^2$$

$$v(t) = -32t$$

$$\text{speed} = 32t$$

Speed reaches 48 ft/sec when

$$32t = 48$$

$$t = \frac{48}{32} = \boxed{\frac{3}{2} \text{ sec}}.$$

Height at $t = 3/2$ is

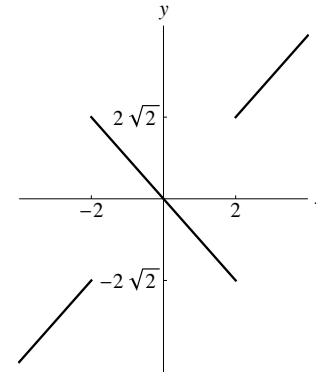
$$\begin{aligned} s\left(\frac{3}{2}\right) &= 81 - 16\left(\frac{3}{2}\right)^2 \\ &= 81 - 16\left(\frac{9}{4}\right) = 81 - 36 \\ &= \boxed{45 \text{ ft}}. \end{aligned}$$

(b) Draco hits the ground when

$$\begin{aligned} s(t) &= 0 \\ 81 - 16t^2 &= 0 \\ 16t^2 &= 81 \\ t^2 &= \frac{81}{16} \\ t &= \boxed{\frac{9}{4} \text{ sec}}. \end{aligned}$$

His velocity at the moment of impact is

$$v\left(\frac{9}{4}\right) = -32\left(\frac{9}{4}\right) = \boxed{-72 \text{ ft/sec}}.$$



Extra Credit

Note that $\frac{x}{|x|} = 1$ for $x > 0$ and $\frac{x}{|x|} = -1$ for $x < 0$.

Similarly $\frac{x^2 - 4}{|x^2 - 4|} = 1$ when $x^2 - 4 > 0$ and $\frac{x^2 - 4}{|x^2 - 4|} = -1$ when $x^2 - 4 < 0$.

We examine the LH and RH limits at $x = -2$ and $x = 2$ to determine whether there is a removable discontinuity or jump discontinuity at those points.

$$\begin{aligned} \lim_{x \rightarrow -2^-} \frac{\sqrt{2}x(x^2 - 4)}{|x^2 - 4|} &= \lim_{x \rightarrow -2^-} \sqrt{2}x = -2\sqrt{2} \\ \lim_{x \rightarrow -2^+} \frac{\sqrt{2}x(x^2 - 4)}{|x^2 - 4|} &= \lim_{x \rightarrow -2^+} -\sqrt{2}x = 2\sqrt{2} \\ \lim_{x \rightarrow 2^-} \frac{\sqrt{2}x(x^2 - 4)}{|x^2 - 4|} &= \lim_{x \rightarrow 2^-} -\sqrt{2}x = -2\sqrt{2} \\ \lim_{x \rightarrow 2^+} \frac{\sqrt{2}x(x^2 - 4)}{|x^2 - 4|} &= \lim_{x \rightarrow 2^+} \sqrt{2}x = 2\sqrt{2} \end{aligned}$$

Because there are jump discontinuities at $x = -2$ and $x = 2$, the function f cannot be extended to be continuous.

Note that $f(x)$ can be rewritten as

$$f(x) = \begin{cases} \sqrt{2}x, & x < -2 \\ -\sqrt{2}x, & -2 < x < 2 \\ \sqrt{2}x, & x > 2 \end{cases}$$