APPM 1340 Midterm 3 Solutions

1a) $x^{3/2} + 2y^{3/2} = 17 \quad (1,4)$

$$\tfrac{3}{2}x^{1/2} + 3y^{1/2}\frac{dy}{dx} = 0$$

$$3y^{1/2}\frac{dy}{dx} = -\frac{3}{2}x^{1/2}$$

$$\frac{dy}{dx} = \frac{-3/2\,x^{1/2}}{3y^{1/2}} = \frac{-x^{1/2}}{2y^{1/2}} = \boxed{\frac{-\sqrt{x}}{2\sqrt{y}}}$$

At $(1,4)$, $\displaystyle \frac{dy}{dx} = \frac{-\sqrt{1}}{2\sqrt{4}} = \frac{-1}{2\cdot 2} = \boxed{\frac{-1}{4}}$

1b) $y^4 = y^2 - x^2 \quad \left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$

$$4y^3\frac{dy}{dx} = 2y\frac{dy}{dx} - 2x$$

$$4y^3\frac{dy}{dx} - 2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx}\left(4y^3 - 2y\right) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{4y^3 - 2y} = \boxed{\frac{-x}{2y^3 - y}}$$

At $\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$, $\displaystyle \frac{dy}{dx} = \frac{-\sqrt{3}/4}{2\left(\frac{\sqrt{3}}{2}\right)^3 - \frac{\sqrt{3}}{2}} = \frac{-\sqrt{3}/4}{2\left(\frac{3\sqrt{3}}{8}\right) - \frac{\sqrt{3}}{2}}$

$$= \frac{-\sqrt{3}/4}{\frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{2}} = \frac{-\sqrt{3}}{3\sqrt{3} - 2\sqrt{3}} = \frac{-\sqrt{3}}{\sqrt{3}} = \boxed{-1}$$

1c) $x^2\cos^2 y - \sin y = 0 \qquad (0,\pi)$

$x^2 \cdot [\cos^2 y]' + \cos^2 y\,[x^2]' - \cos y \frac{dy}{dx} = 0$

$x^2\left(2(\cos y)(-\sin y)\frac{dy}{dx}\right) + (\cos^2 y)(2x) - \cos y \frac{dy}{dx} = 0$

$-2x^2 \sin y \cos y \frac{dy}{dx} - \cos y \frac{dy}{dx} = -2x\cos^2 y$

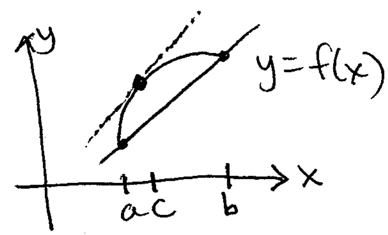
$\frac{dy}{dx}\left(2x^2 \sin y \cos y + \cos y\right) = 2x\cos^2 y$

$$\frac{dy}{dx} = \boxed{\frac{2x\cos^2 y}{2x^2 \sin y \cos y + \cos y}}$$

At $(0,\pi)$, $\frac{dy}{dx} = \frac{2(0)(-1)^2}{2(0)(0)(-1)-1} = \frac{0}{0-1} = \boxed{0}$

2a) Given a function f that is continuous on $[a,b]$ and differentiable on (a,b), there is at least one point c in (a,b) at which

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$



average rate of change over the interval equals the instantaneous rate of change at a point in the interval.

2b) $f(x) = \sqrt{2x-1}$ on $[1,3]$

f is continuous on $[1,3]$ because f is defined for all $x \geq \frac{1}{2}$.

f is differentiable on $[1,3]$ because $f'(x) = \frac{2}{2\sqrt{2x-1}} = \frac{1}{\sqrt{2x-1}}$ is defined for all $x > \frac{1}{2}$.

We wish to find c such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{\sqrt{2c-1}} = \frac{f(3)-f(1)}{3-1} = \frac{\sqrt{5}-1}{2}$$

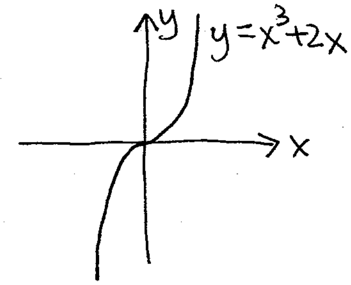
$$\sqrt{2c-1} = \frac{2}{\sqrt{5}-1}$$

$$2c-1 = \left(\frac{2}{\sqrt{5}-1}\right)^2$$

$$2c = \left(\frac{2}{\sqrt{5}-1}\right)^2 + 1$$

$$c = \frac{1}{2}\left(\left(\frac{2}{\sqrt{5}-1}\right)^2 + 1\right) = \frac{1}{2}\left(\frac{2}{3-\sqrt{5}} + 1\right) = \boxed{\frac{1}{3-\sqrt{5}} + \frac{1}{2}}$$

2c) $f(x) = x^3 + 2x$
$f'(x) = 3x^2 + 2$



$f'(x) = 0$ has no solutions
because $3x^2+2=0 \Rightarrow 3x^2=-2 \Rightarrow x^2 = \frac{-2}{3}$

$f'(x) = 3x^2+2$ is defined for all x.

Therefore f' has no critical points. It is positive for all x so $f(x)$ is an increasing function throughout its domain $(-\infty, \infty)$. We conclude that f has no local maxima or minima.

3a) $y = x^{2/3}$

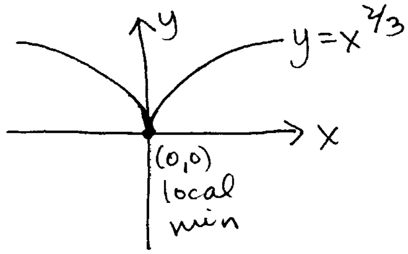
$y' = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$ critical: y' undefined at $x=0$

$y'' = \frac{-2}{9}x^{-4/3} = \frac{-2}{9\sqrt[3]{x^4}}$ critical: y'' undefined at $x=0$

y': $-$ | $+$
$\searrow$ 0 $\nearrow$

y'': $-$ | $-$
down 0 down

summary
y': $\searrow$ $\nearrow$
y'': $\cap$ $\cap$
0



y has a cusp at (0,0)
but no inflection pts.

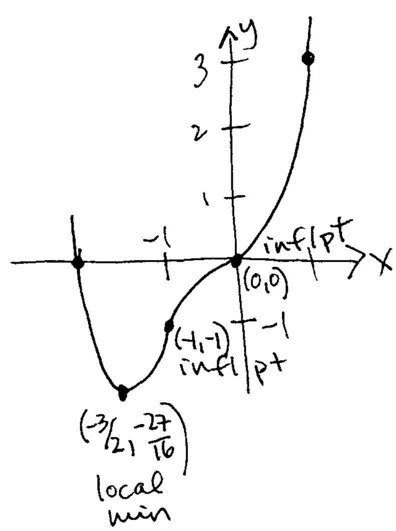
3b) $y = x^4 + 2x^3 = x^3(x+2)$

$y' = 4x^3 + 6x^2 = 2x^2(2x+3)$ critical: $y'=0$ at $x = 0, -3/2$

$y'' = 12x^2 + 12x = 12x(x+1)$ critical: $y''=0$ at $x=0,-1$

y': $-$ | $+$ | $+$
$\searrow$ $-3/2$ $\nearrow$ 0 $\nearrow$

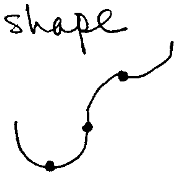
y'': $+$ | $-$ | $+$
up -1 down 0 up



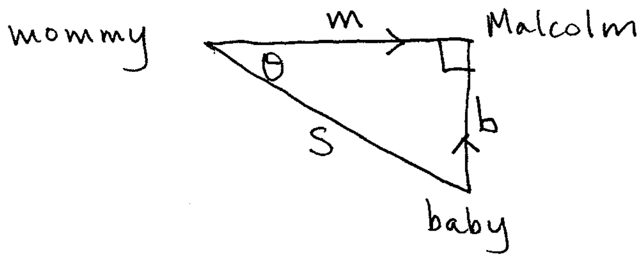
summary

y': $\searrow$ $\nearrow$ $\nearrow$ $\nearrow$
y'': $\cup$ $\cup$ $\cap$ $\cup$
$-3/2$ -1 0

general shape



4a)



Find $\frac{dm}{dt}$ when $b = 50\,m$, $s = 130\,m$, $\frac{ds}{dt} = -26\,m/sec$

given $\frac{db}{dt} = -10\,m/sec$

When $b = 50$, $s = 130$, then $m = \sqrt{130^2 - 50^2} = \sqrt{14400} = 120\,m$.

(Pythagorean triple: $5^2 + 12^2 = 13^2$.)

Equation: $s^2 = m^2 + b^2$

Differentiate: $2s\frac{ds}{dt} = 2m\frac{dm}{dt} + 2b\frac{db}{dt}$.

$$s\frac{ds}{dt} = m\frac{dm}{dt} + b\frac{db}{dt}.$$

$$(130)(-26) = (120)\frac{dm}{dt} + (50)(-10)$$

$$\frac{dm}{dt} = \frac{-(130)(26) + (50)(10)}{120} = \frac{-3380+500}{120}$$

$$= \frac{-2880}{120} = -24$$

The mommy raptor is moving at $\boxed{24 m/sec}$.

4b) Equation: $\sin\theta = \frac{b}{s}$ We wish to find $\frac{d\theta}{dt}$.

Differentiate:

$$\cos\theta\frac{d\theta}{dt} = \frac{s\frac{db}{dt} - b\frac{ds}{dt}}{s^2}$$

$$\left(\frac{120}{130}\right)\frac{d\theta}{dt} = \frac{(130)(-10) - (50)(-26)}{(130)^2}$$

$$\frac{d\theta}{dt} = \left(\frac{130}{120}\right)\left(\frac{-1300+1300}{(130)^2}\right) = \boxed{0 \text{ rad/sec}}$$

(Also acceptable eqs:

$\tan\theta = \frac{b}{m}$

$\cos\theta = \frac{m}{s}$

- ...)

The proportions of the right triangle stay the same as the raptors approach Malcolm.

4c) $D = RT \Rightarrow \text{Time} = \dfrac{\text{Distance}}{\text{Rate}}$

$\text{Time}_{\text{Baby}} = \dfrac{50\text{m}}{10\text{ m/sec}} = 5\text{ sec}$ to reach Malcolm

$\text{Time}_{\text{mommy}} = \dfrac{120\text{m}}{24\text{ m/sec}} = 5\text{ sec}$ " " "

Mommy and baby reach Malcolm at the <u>same time</u>.

Alternate explanation:

Since $\dfrac{d\theta}{dt} = 0$, the proportions of the right triangle stay the same up until the moment the two raptors reach Malcolm.

(Raptor problem inspired by the comic "Substitute" at www.xkcd.com/135/.)