1. a) $\lim_{x\to\infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x\to\infty} \frac{2+7/x^3}{1-1/x+1/x^2+7/x^3} = \frac{2+0}{1-0+0+0} = \boxed{2.}$

b) $\frac{d}{dt}\sqrt{1+\cos(t^2)} = \frac{1}{2\sqrt{1+\cos(t^2)}}\cdot\frac{d}{dt}\left(1+\cos(t^2)\right) = \frac{-\sin(t^2)}{2\sqrt{1+\cos(t^2)}}\cdot\frac{d}{dt}(t^2)$

$\Rightarrow \boxed{\frac{dy}{dt} = \frac{-t\sin(t^2)}{\sqrt{1+\cos(t^2)}}}$

c) $\frac{d}{d\theta}\sin(\frac{\pi}{3}) = \boxed{0}$ since $\sin(\pi/3)$ is a constant!

2 a) If $f(x)$ is continuous on $[a,b]$ & differentiable on (a,b) then there exists at least one point $c\in(a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

b) $f(x) = x^{4/3}$ is continuous on $[-1,8]$ & $f'(x) = \frac{4}{3}x^{1/3}$ is defined $\forall x$ so f is differentiable on $(-1,8)$ $\Rightarrow$ $\boxed{\text{YES}}$ the MVT applies.

3 a) $\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(16) \Rightarrow 3x^2+3y^2\frac{dy}{dx} = 0 \Rightarrow \boxed{\frac{dy}{dx} = \frac{-x^2}{y^2}}$

b) $x^2+y^2\frac{dy}{dx} = 0 \Rightarrow \frac{d}{dx}\left(x^2+y^2\frac{dy}{dx}\right) = \frac{d}{dx}(0)$

$\Rightarrow 2x + \left(2y\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) + y^2\frac{d^2y}{dx^2} = 0$

$\Rightarrow \boxed{\frac{d^2y}{dx^2} = \frac{-2}{y^2}\left[x+y\left(\frac{dy}{dx}\right)^2\right]}$

OR, equivalently, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-x^2}{y^2}\right) = \frac{-1}{y^4}\left[2xy^2 - x^2 2y\frac{dy}{dx}\right]$

$\Rightarrow \boxed{\frac{d^2y}{dx^2} = \frac{-2x}{y^3}\left[y - x\frac{dy}{dx}\right]}$

(note: $\frac{-2}{y^2}\left[x+y\left(\frac{dy}{dx}\right)^2\right] = \frac{-2}{y^2}\left[x - \frac{x^2}{y}\frac{dy}{dx}\right]$ since $\frac{dy}{dx} = \frac{-x^2}{y^2}$

$= \frac{-2x}{y^2}\left[1-\frac{x}{y}\frac{dy}{dx}\right] = \frac{-2x}{y^3}\left[y-x\frac{dy}{dx}\right]$ ✓)

3 c) At $(x,y) = (2,2)$ $\quad \frac{dy}{dx} = -\left(\frac{2}{2}\right)^2 \Rightarrow \boxed{\frac{dy}{dx} = -1}$

Then $\frac{d^2y}{dx^2} = \frac{-2}{2^2}\left[2 + 2 \times 1\right] = \frac{-8}{4} \Rightarrow \boxed{\frac{d^2y}{dx^2} = -2}$

d) The tangent line is $y = mx + c$ where $m = \left.\frac{dy}{dx}\right|_{(2,2)} = -1$

$\Rightarrow y = -x + c$

The line goes through $(x,y) = (2,2)$ so $2 = -2 + c \Rightarrow c = 4$

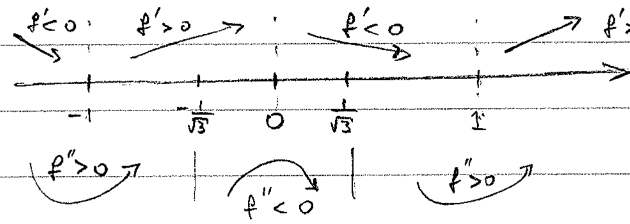
$\Rightarrow$ the equation of the tangent line is $\boxed{y = 4 - x}$

4. $f(x) = 2x^4 - 4x^2 + 1 \quad \Rightarrow f'(x) = 8x^3 - 8x = 8x(x^2 - 1)$

$\Rightarrow$ critical points are $x = 0, \pm 1$

$f''(x) = 24x^2 - 8 = 8(3x^2 - 1)$

$\Rightarrow f''(x) = 0$ when $x = \pm \frac{1}{\sqrt{3}}$



$\lim_{x \to \pm\infty} f(x) = +\infty$ so no asymptotes

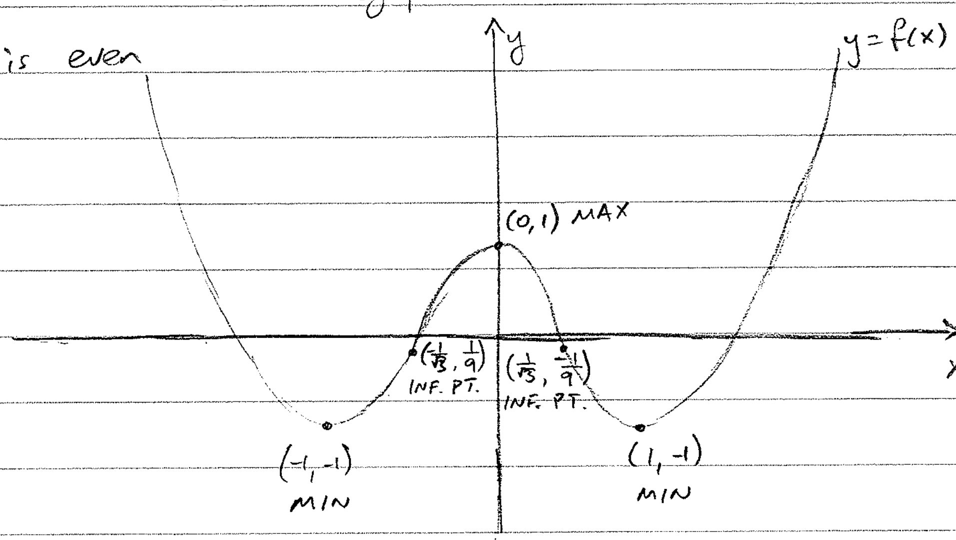
Local minima at $x = \pm 1 \quad \Rightarrow f(x) = 2 - 4 + 1 = -1$

Local maximum at $x = 0 \quad \Rightarrow f(x) = 1$. $\quad$ Inflection pts at $x = \mp\frac{1}{\sqrt{3}}$

f' defined everywhere $\Rightarrow$ no cusps $\quad\quad \Rightarrow f(x) = \frac{2}{9} - \frac{4}{3} + 1 = \frac{-1}{9}$

f " " $\Rightarrow$ no vertical asymptotes

$f(x) = f(-x)$ so f is even

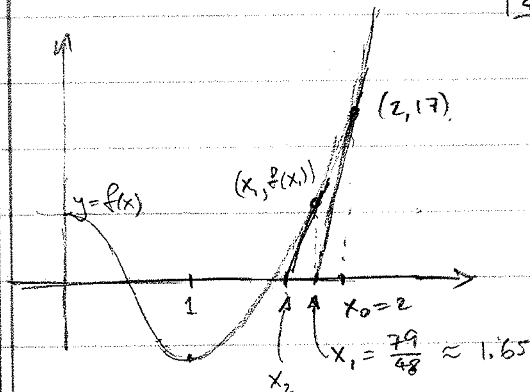


5 a) From 4, $f' = 8x(x^2-1)$

Newton's Method: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{32-16+1}{16(4-1)} = 2 - \frac{17}{48}$

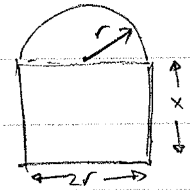
$= \boxed{\frac{79}{48}}$

b)



6. Perimeter $= \pi r + 2r + 2x$

Transmitted light $\propto L(r,x) = \frac{1}{4}\pi r^2 + 2rx$



Perimeter is fixed $\Rightarrow 2x = P - (2+\pi)r$ where P = constant

$\Rightarrow L = L(r) = \frac{\pi}{4}r^2 + r(P-(2+\pi)r) = \frac{\pi}{4}r^2 + Pr - (2+\pi)r^2$

$= Pr - (2+\frac{3\pi}{4})r^2$

$\Rightarrow L' = P - (4+\frac{3\pi}{2})r$ so $L'=0 \Rightarrow P = \left(\frac{8+3\pi}{2}\right)r \Rightarrow \boxed{r = \frac{2P}{8+3\pi}}$

Then $2x = P - (2+\pi)r = P - \frac{2(2+\pi)}{8+3\pi}P = \frac{8+3\pi-4-2\pi}{8+3\pi}P = \frac{4+\pi}{8+3\pi}P$

$\Rightarrow \boxed{x = \frac{4+\pi}{2(8+3\pi)}P}$

The height-to-width ratio of the window should be

$\boxed{\frac{x}{2r} = \frac{4+\pi}{8}}$

Note: $L'' = -(4+\frac{3\pi}{2}) < 0$ so these dimensions give <u>max</u> light transmitted (not min.)