

1 Measuring the Earth

Around 240 BC, Eratosthenes make an attempt to measure the radius of the Earth (and was very successful, given his simple method). He measured the length of the shadow of a stick at noon on the summer solstice in the city of Alexandria and calculated the radius of the Earth based on the fact that the sun is directly overhead on that day in a city 5000 stades (about 530 miles) away.

One way to do this would be to use the formula

$$\tan\left(\frac{x}{R}\right) = \frac{S}{L}$$

where x is the distance between the cities, R is the radius of the Earth (what you're trying to calculate), S is the length of the shadow of the stick, which has length L .

2 Errors

Q: Assuming S and L can be measured very accurately, how do errors in the measurement of x affect the calculation of R ?

A: We assume that $R = R(x)$ and so, effectively, S and L are constants. Differentiating implicitly, we get

$$\begin{aligned}\frac{d}{dx} \left[\tan\left(\frac{x}{R}\right) \right] &= \frac{d}{dx} \left[\frac{S}{L} \right] \\ \Rightarrow \sec^2\left(\frac{R - x \frac{dR}{dx}}{R^2}\right) &= 0 \\ \Rightarrow R - x \frac{dR}{dx} &= 0 \\ \Rightarrow \frac{dR}{dx} &= \frac{R}{x}\end{aligned}$$

From this we have that

$$\Delta R \approx \frac{dR}{dx} \Delta x = \frac{R}{x} \Delta x$$

In terms of percentage errors, this gives

$$\frac{\Delta R}{R} \approx \frac{\Delta x}{x}$$

In words: the percentage error in the calculated value of R is the same as the percentage error in the measurement of x .

3 Linearization

The above implicit equation for R is pretty messy to work with. Maybe we could approximate it somehow...

Q: Use an appropriate linearization to simplify the equation $\tan(x/R) = S/L$.

A: We want to linearize the tangent term, so we need to know what the linearization of $\tan(y)$ is, about some appropriate value of y . We may assume that $x \ll R$, so x/R is very small; hence, we will use $y = 0$ as the point about which to linearize $\tan(y)$. Since $\tan(0) = 0$, and $\frac{d}{dy} \tan(y) = \sec^2(y)$ and $\sec^2(0) = 1$, the linearization of $\tan(y)$ is

$$\tan(y) \approx 0 + 1(y - 0) = y$$

Hence, the approximate version of the original equation is

$$\tan\left(\frac{x}{R}\right) \approx \frac{x}{R} = \frac{S}{L} \Leftrightarrow R = x \frac{L}{S}$$