APPM 1350: Section 1.4: Extensions of the Limit Concept

Class exercize

(#45, p.66) It can be shown that

$$1 - \frac{x^2}{6} < \frac{x\sin x}{2-2\cos x} < 1$$

hold for all values of x close to zero. What if anything does this tell you about

$$\lim_{x\to 0} \frac{x\sin x}{2-2\cos x} \quad ?$$

Solution:

Note that if

$$a < b < c \qquad \text{(i.e., strict inequality)}$$

then

$$a \leq b \leq c$$

also. So

$$1 - \frac{x^2}{6} < \frac{x\sin x}{2-2\cos x} < 1$$

$$\Rightarrow \quad 1 - \frac{x^2}{6} \leq \frac{x\sin x}{2-2\cos x} \leq 1$$

$$\Rightarrow \lim_{x\to 0}\left(1 - \frac{x^2}{6}\right) \leq \lim_{x\to 0}\left(\frac{x\sin x}{2-2\cos x}\right) \leq \lim_{x\to 0} 1$$

$$\Rightarrow \quad 1 \leq \lim_{x\to 0}\left(\frac{x\sin x}{2-2\cos x}\right) \leq 1$$

This can only be true if the equality holds, i.e.,

$$\lim_{x\to 0} \frac{x\sin x}{2-2\cos x} = 1$$

Class exercize

Evaluate

$$\lim_{x \to 2} \frac{\sqrt{x^2+5}-3}{x^2-2x}$$

Solution:

(denominator is zero at $x=2$)

$$\lim_{x \to 2} \frac{\sqrt{x^2+5}-3}{x^2-2x} = \lim_{x \to 2} \frac{(\sqrt{x^2+5}-3)(\sqrt{x^2+5}+3)}{(x^2-2x)(\sqrt{x^2+5}+3)}$$

$$= \lim_{x \to 2} \frac{x^2+5-9}{(x^2-2x)(\sqrt{x^2+5}+3)}$$

$$= \lim_{x \to 2} \frac{x^2-4}{(x^2-2x)(\sqrt{x^2+5}+3)}$$

$$= \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{x\cancel{(x-2)}(\sqrt{x^2+5}+3)}$$

$$= \lim_{x \to 2} \frac{x+2}{x(\sqrt{x^2+5}+3)}$$

$$= \frac{4}{2(3+3)}$$

$$= \frac{1}{3}$$

Infinite Limits

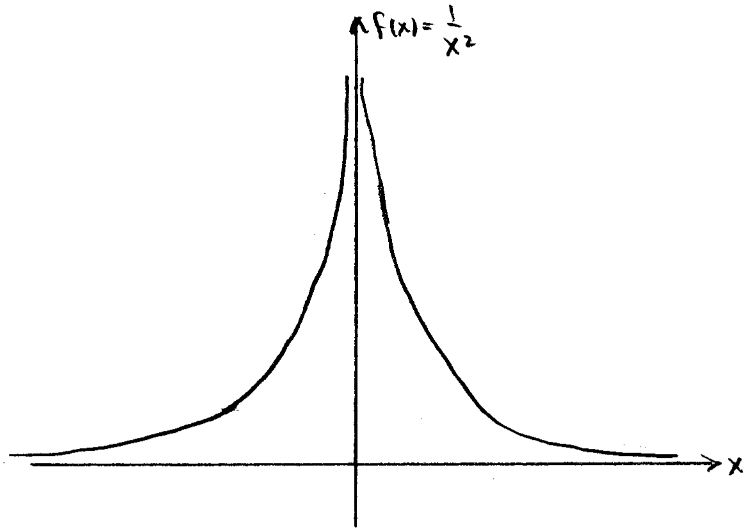
Up till now we have dealt with limits of functions where the limits are finite. For example,

$$\lim_{x \to 2} x^2 = 4$$

Now we extend this concept to where the function "blows up" or becomes infinite in the limit. So consider, for example, the function

$$f(x) = \frac{1}{x^2} \qquad (x \neq 0)$$

The function exists and has a finite value for all x <u>except</u> $x=0$.



However, as x gets closer and closer to zero, $\frac{1}{x^2}$ gets larger and larger. If the function was defined at $x=0$ its value would in effect be infinite. It is not defined at $x=0$, but we can say

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Its limit exists and is infinite.

Example

$$\lim_{x \to 0} \frac{1}{1-\cos x} = \infty$$

Example

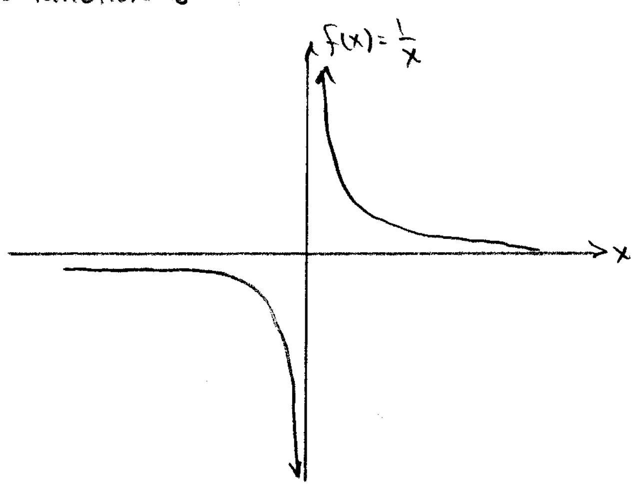
$$\lim_{x \to 0} \frac{1}{\cos x - 1} = -\infty$$

One-Sided Limits

Now consider

$$f(x) = \frac{1}{x} \qquad (x \neq 0)$$

The graph of this function is



In this case the function "goes to infinity" as $x \to 0$ from the right (i.e, $x > 0$) but "goes to negative infinity" as $x \to 0$ from the left (i.e, $x < 0$). The limit at $x = 0$ does not exist because we get different values (even infinite ones) depending on what direction we approach $x = 0$. However, if we consider the case of approaching $x = 0$ from just one side at a time we can say something about the behavior of the function at $x = 0$. In particular, the closer and closer we get to

$x=0$ from the left (i.e., $x<0$) the more negative $f(x)=\frac{1}{x}$ becomes. The function in fact approaches $-\infty$ as $x\to 0$ from the left. In this case we say

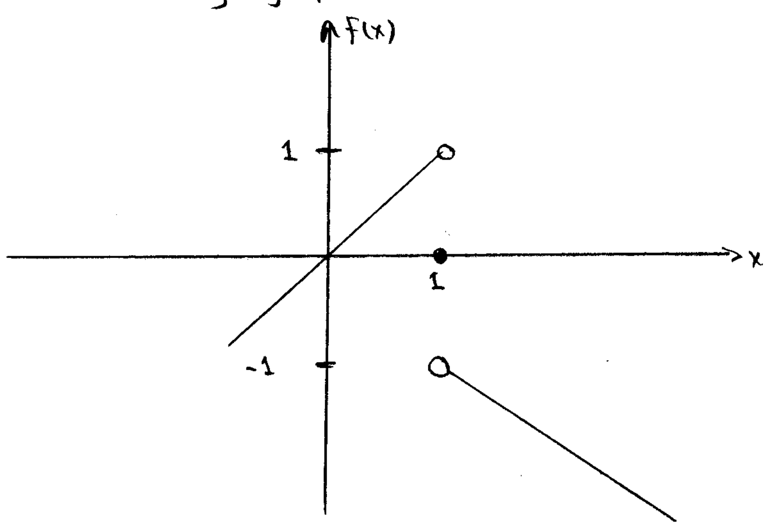
$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

The $x\to 0^-$ means we only approach $x=0$ from the left. By the same token we can say

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

where $x\to 0^+$ means we approach x from the right, These are examples of <u>one-sided limits</u>.

Let's consider a less exotic example. Consider the function shown in the following graph:



Here we would approach the value 1 as $x\to 1$ from the left but approach the value -1 as $x\to 1$ from the right (note that $f(x)=0$ at $x=1$ itself). We would then write:

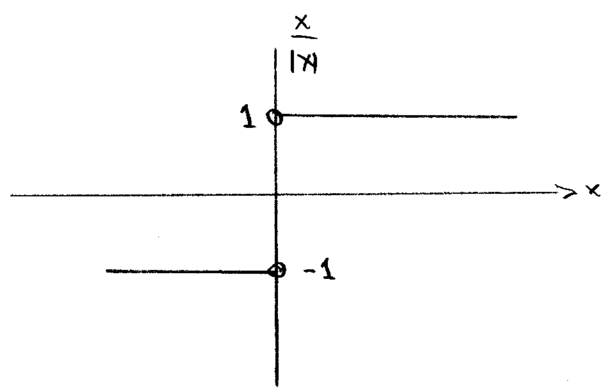
$$\lim_{x \to 1^-} f(x) = 1 \qquad \text{(left-handed limit)}$$

$$\lim_{x \to 1^+} f(x) = -1 \qquad \text{(right-handed limit)}$$

Class exercize

Find the one-sided limits of $f(x) = \dfrac{x}{|x|}$ as $x \to 0$

Solution:



$f(x)$ does not exist at $x=0$. From the graph (or algebraically):

$$\lim_{x \to 0^-} \frac{x}{|x|} = -1 \qquad \text{(left-hand limit)}$$

$$\lim_{x \to 0^+} \frac{x}{|x|} = 1 \qquad \text{(right-hand limit)}$$

But note;

$$\lim_{x \to 0} \frac{x}{|x|}$$

does not exist! For the true limit to exist, the left- and right-hand limits must both exist and be equal.

Informal definition of left- and right-hand limits: Let $f(x)$ be defined on an interval (a,b) where $a<b$. If $f(x)$ approaches arbitrarily close to L as x approaches a from within that interval, then we say that f has a right-hand limit L at a and write

$$\lim_{x \to a^+} f(x) = L$$

Let $f(x)$ be defined on an interval (c,a) where $c<a$. If $f(x)$ approaches arbitrarily close to M as x approaches a from within that interval, then we say f has a left-hand limit M at a and write

$$\lim_{x \to a^-} f(x) = M$$

We can extend this definition to "infinite" limits by saying $L=\infty$ if $f(x)$ gets larger and larger as $x \to a^+$, or $L=-\infty$ if $f(x)$ gets larger and larger with negative values as $x \to a^+$. Same idea for M.

But, the true limit only exists if $L=M$!

Thm: A function $f(x)$ has a limit as $x \to c$ if and only if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

In this case we write

$$\lim_{x \to c} f(x) = L$$

Example

Consider $f(x) = x^2$. Then

$$\lim_{x \to 2^-} x^2 = 4$$

$$\lim_{x \to 2^+} x^2 = 4$$

$$\Rightarrow \lim_{x \to 2} x^2 = 4$$

Example

Consider $f(x) = \csc x$. Then

$$\lim_{x \to 0^-} \csc x = \lim_{x \to 0^-} \frac{1}{\sin x} = -\infty$$

$$\lim_{x \to 0^+} \csc x = \lim_{x \to 0^+} \frac{1}{\sin x} = +\infty$$

The true $\lim_{x \to 0} \csc x$ <u>does not exist</u> because the left- and right-hand limits have different values. But the left- and right-hand limits do exist.

Example

Consider $f(x) = \dfrac{x+2}{x^2-4}$

$$\lim_{x\to 2^-} \frac{x+2}{x^2-4} = \lim_{x\to 2^-} \frac{(x+2)}{(x+2)(x-2)}$$

$$= \lim_{x\to 2^-} \frac{1}{x-2}$$

$$= -\infty$$

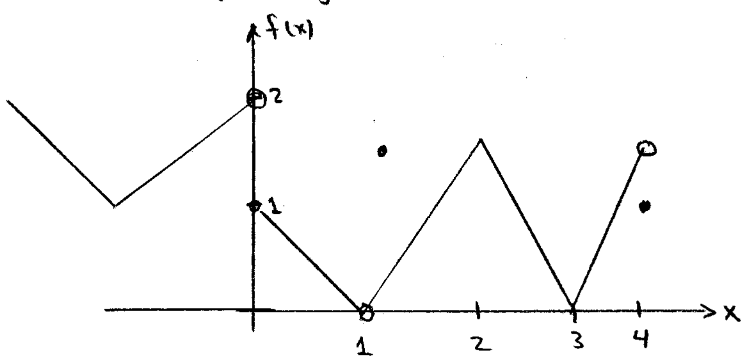
$$\lim_{x\to 2^+} \frac{x+2}{x^2-4} = \lim_{x\to 2^+} \frac{1}{x-2}$$

$$= +\infty$$

$\lim_{x\to 2} \dfrac{x+2}{x^2-4}$ does not exist.

Example

$f(x)$ is defined by the graph:



a) $\lim_{x\to 0^-} f(x) = 2$

$\lim_{x\to 0^+} f(x) = 1$

$\lim_{x\to 0} f(x)$ doesn't exist

b) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 0$ despite $f(1) = \frac{3}{2}$.

c) $\lim_{x\to 4^-} f(x) = \frac{3}{2}$ but $\lim_{x\to 4^+} f(x)$ doesn't exist because $f(x)$ is not defined for $x > 4$. Similarly, $\lim_{x\to 4} f(x)$ doesn't exist.

Class exercize

(#42, p 85) Find $\lim \dfrac{x^2-3x+2}{x^3-4x}$ as

a) $x \to 2^+$
b) $x \to -2^+$
c) $x \to 0^-$
d) $x \to 1^+$
e) What, if anything, can be said about the true limit as $x \to 0$?

Solution:

Note: $$f(x) = \frac{x^2-3x-2}{x^3-4x}$$

$$= \frac{(x-2)(x-1)}{x(x^2-4)}$$

$$= \frac{(x\cancel{-2})(x-1)}{x(x\cancel{-2})(x+2)}$$

$$= \frac{x-1}{x(x+2)}$$

So: a) $\lim\limits_{x \to 2^+} f(x) = \dfrac{1}{2(4)} = \dfrac{1}{8}$

b) $\lim\limits_{x \to -2^+} f(x) = \infty$

As $x \to 2^+$ from the right, $x-1<0$, $x<0$, but $x+2>0$. So we have $\dfrac{\text{negative}}{(\text{negative})(\text{positive})} = \text{positive}$

c) $\lim\limits_{x \to 0^-} f(x) = \infty$

d) $\lim\limits_{x \to 1^+} f(x) = 0$

e) $\lim\limits_{x \to 0^+} f(x) = -\infty$. Since $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$, then

the $\lim\limits_{x \to 0} f(x)$ does not exist.

Even More Examples

Example

Prove $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$

Proof:

Consider that for $x \neq 0$,

$$0 \leq \left| x \sin\left(\tfrac{1}{x}\right) \right| \leq |x|$$

$$\Rightarrow \operatorname*{limit}_{x \to 0} 0 \leq \operatorname*{limit}_{x \to 0} \left| x \sin\left(\tfrac{1}{x}\right) \right| \leq \lim_{x \to 0} |x|$$

$$\Rightarrow 0 \leq \lim_{x \to 0} \left| x \sin\left(\tfrac{1}{x}\right) \right| \leq 0$$

$$\Rightarrow \lim_{x \to 0} \left| x \sin\left(\tfrac{1}{x}\right) \right| = 0$$

$$\Rightarrow \left| \operatorname*{limit}_{x \to 0} \left(x \sin \tfrac{1}{x} \right) \right| = 0$$

$$\Rightarrow \lim_{x \to 0} x \sin\left(\tfrac{1}{x}\right) = 0$$

$\blacksquare$

Example

Does the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x < 0 \\ \sin \frac{1}{x} & \text{if } x > 0 \end{cases}$$

have a limit at $x = 0$? At $x = \frac{1}{\pi}$? How about one-sided limits?

Solution:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x \sin \frac{1}{x} = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sin \frac{1}{x} \quad \text{doesn't exist}$$

So there is no limit at $x = 0$.

At $x = \frac{1}{\pi}$:

$$\lim_{x \to \frac{1}{\pi}^-} f(x) = \lim_{x \to \frac{1}{\pi}^-} \sin\left(\frac{1}{x}\right) = 0$$

$$\lim_{x \to \frac{1}{\pi}^+} f(x) = \lim_{x \to \frac{1}{\pi}^+} \sin\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow \lim_{x \to \frac{1}{\pi}} f(x) = 0$$

(left- and right-hand limits exist and are equal $\Rightarrow$ true limit exists)

Example

Evaluate $\lim_{x \to 0} x \cot 2x$

Solution:

$$\lim_{x \to 0} x \cot 2x = \lim_{x \to 0} x \frac{\cos 2x}{\sin 2x}$$

$$= \lim_{x \to 0} \frac{x}{\sin 2x} (\cos 2x)$$

$$= \lim_{x \to 0} \frac{x \ \cos 2x}{2 \sin x \cos x} \qquad \text{since } \sin 2x = 2 \sin x \cos x$$

$$= \frac{1}{2} \lim_{x \to 0} \left(\frac{x}{\sin x}\right)\left(\frac{\cos 2x}{\cos x}\right)$$

$$= \frac{1}{2} \lim_{x \to 0} \left(\frac{x}{\sin x}\right) \frac{\lim_{x \to 0} \cos 2x}{\lim_{x \to 0} \cos x}$$

$$= \frac{1}{2} \cdot 1 \cdot \frac{1}{1}$$

$$= \frac{1}{2}$$

where we used $\lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{\sin x}{x} = 1$ which I proved in my lecture notes on the sandwich theorem (the one that covered both sections 1.1 and 1.2).