APPM 1350: Forgiveness Exam 3: Solutions

1a) Estimate the integral $\int_0^3 \sqrt{x+1}\,dx$ using $n=6$ rectangles and left-hand endpoints.

Soln:

n	x_n	$f(x_n)$
0	0	1
1	$1/2$	$\sqrt{3/2}$
2	1	$\sqrt{2}$
3	$3/2$	$\sqrt{5/2}$
4	2	$\sqrt{3}$
5	$5/2$	$\sqrt{7/2}$
6	3	2

$\Delta x = \frac{3}{6} = \frac{1}{2}$

$$\boxed{\int_0^3 \sqrt{x+1}\,dx \approx \frac{1}{2}\left[1+\sqrt{\tfrac{3}{2}}+\sqrt{2}+\sqrt{\tfrac{5}{2}}+\sqrt{3}+\sqrt{\tfrac{7}{2}}\right]}$$

b) Estimate the same integral using the trapezoidal sum for $n=6$.

Soln:
$$\boxed{\int_0^3 \sqrt{x+1}\,dx \approx \frac{1}{4}\left[1+2\sqrt{\tfrac{3}{2}}+2\sqrt{2}+2\sqrt{\tfrac{5}{2}}+2\sqrt{3}+2\sqrt{\tfrac{7}{2}}+2\right]}$$

c) How large do you have to make n to be sure that the corresponding trapezoidal estimate is within 10^{-4} of the real value of the integral?

Soln:
$$\frac{b-a}{2} h^2 M \leq 10^{-4} \qquad M = \max_{x\in[0,3]} |f''(x)| = \frac{1}{4}$$

$$\Rightarrow \frac{3^3}{2}\,\frac{1}{n^2}\,\frac{1}{4} \leq 10^{-4}$$

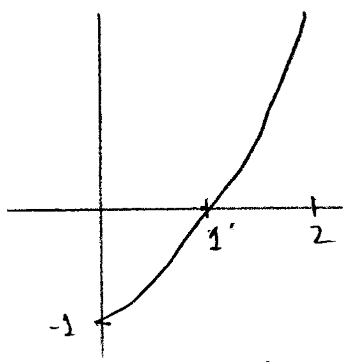
$$\Rightarrow n^2 \geq \frac{27}{8}10^4$$

$$\Rightarrow n \geq \sqrt{\frac{27}{8}} \times 100$$

$$\Rightarrow \boxed{n \geq 200}$$ (rounding to next integer)

2. Find the total area of the two regions enclosed between the graph of the function $y = x^2 - 1$ on the interval $[0, 2]$ and the x axis.

Soln:



$$\text{Area} = \int_0^2 |x^2-1|\,dx = -\int_0^1 (x^2-1)\,dx + \int_1^2 (x^2-1)\,dx$$

$$= -\left(\frac{x^3}{3} - x\right)\Big|_0^1 + \left(\frac{x^3}{3} - x\right)\Big|_1^2$$

$$= -\left(\frac{1}{3} - 1\right) + \left(\frac{8}{3} - 2 - \frac{1}{3} + 1\right)$$

$$= \frac{2}{3} + \left(\frac{7}{3} - 1\right)$$

$$= \frac{2}{3} + \frac{4}{3}$$

$$= \boxed{2}$$

3. Calculate the integrals:

a) $\int \frac{\sqrt[3]{t}+1}{\sqrt{t}}\,dt$

Soln: $\int \frac{t^{1/3}+1}{t^{1/2}}\,dt = \int (t^{-1/6} + t^{-1/2})\,dt$

$$= \frac{t^{5/6}}{(5/6)} + \frac{t^{1/2}}{(\frac{1}{2})} + c$$

$$= \boxed{\frac{6t^{5/6}}{5} + 2t^{1/2} + c}$$

b) $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx = 2\int u^3\,du$

Let $u = 1+\sqrt{x}$

$du = \frac{dx}{2\sqrt{x}}$

$$= \frac{u^4}{2} + c$$

$$= \boxed{\frac{1}{2}(1+\sqrt{x})^4 + c}$$

c) $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt = \frac{1}{2}\int_1^8 u^{1/3}\,du$

Let $u = t^2+1$

$du = 2t\,dt$

$$= \frac{1}{2}\left.\frac{u^{4/3}}{(4/3)}\right|_1^8$$

$$= \frac{3}{8}\left(8^{4/3} - 1\right)$$

$$= \frac{3}{8}(2^4 - 1)$$

$$= \frac{3}{8}(15)$$

$$= \boxed{\frac{45}{8}}$$

4. a) State the Fundamental Theorem of Calculus

Soln:

Part 1: Assume f is continuous on $[a,b]$. Then $F(x) = \int_a^x f(t)\,dt$ has a derivative at every $x \in [a,b]$ and

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x) \qquad x \in [a,b]$$

Part 2: Assume f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$

b) Calculate the derivative of the function $h(x) = \int_0^{\sqrt{x}} \cos\theta \, d\theta$

Soln: $$\frac{d}{dx} \int_0^{\sqrt{x}} \cos\theta\, d\theta = \cos\sqrt{x} \, \frac{d}{dx}\sqrt{x}$$

$$= \boxed{\frac{\cos\sqrt{x}}{2\sqrt{x}}}$$

c) Find the critical points of the function h given in part b.

Soln:

$$\frac{dh}{dx} = 0 \Rightarrow \frac{\cos\sqrt{x}}{2\sqrt{x}} = 0$$

$$\Rightarrow \sqrt{x} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

So $\boxed{x = \left(\frac{\pi}{2}\right)^2, \left(\frac{3\pi}{2}\right)^2, \left(\frac{5\pi}{2}\right)^2, \ldots}$

5. a) Find the average value of the function $f(x) = 3x^2 - 3$ on the interval $[0,1]$.

Soln: $\text{Avg} = \dfrac{1}{1-0} \displaystyle\int_0^1 (3x^2-3)\,dx$

$$= x^3 - 3x \Big|_0^1$$

$$= 1 - 3$$

$$= \boxed{-2}$$

b) The Mean Value Theorem states that there is a value $x = c$ such that $f(c)$ equals the average value of f. Find this value of c.

Soln:

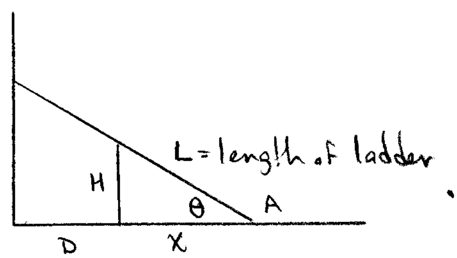
$$3c^2 - 3 = -2$$

$$\Rightarrow 3c^2 = 1$$

$$\Rightarrow \boxed{c = \frac{1}{\sqrt{3}}}$$

Extra Credit A fence of height H is D feet away from a vertical wall. At what angle θ should a ladder be leaned against the fence in order that the minimum length ladder be required to stretch from the ground to the wall?

Soln.



We want to minimize L as a function of θ. So

$$L = (x+D)\sec\theta$$

and

$$x = H\cot\theta$$

$$\Rightarrow L = (H\cot\theta + D)\sec\theta$$

$$= H\cot\theta\sec\theta + D\sec\theta$$

$$= H\csc\theta + D\sec\theta$$

$$\Rightarrow \frac{dL}{d\theta} = H(-\csc\theta\cot\theta) + D(\sec\theta\tan\theta) = 0$$

$$\Rightarrow H(\csc\theta\cot\theta) = D(\sec\theta\tan\theta)$$

$$\Rightarrow \frac{\csc\theta\cot\theta}{\sec\theta\tan\theta} = \frac{D}{H}$$

$$\Rightarrow \frac{1}{\tan^3\theta} = \frac{D}{H}$$

$$\Rightarrow \boxed{\tan\theta = \left(\frac{H}{D}\right)^{1/3}}$$