APPM 1350: Section 2.3: Rates of Change

Recall the following definitions:

Def: The average rate of change of a function $f(x)$ with respect to x over the interval x_0 to $x_0 + h$ is

$$\text{Average rate of change} = \frac{f(x_0+h) - f(x_0)}{h}$$

The instantaneous rate of change of $f(x)$ at x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

provided the limit exists.

When we say "rate of change" what we generally mean is "instantaneous rate of change".

We have encountered numerous examples of rate of change in previous lectures. One type of rate of change of particular importance is when our function gives the position of a body in motion. We now explore this in detail.

Motion Along a Line

Def Let $s(t)$ be the position of a body moving along a straight line as a function of time. The displacement of the object between times t_1 and t_2 is

$$\Delta s = s(t_1) - s(t_2)$$

Over an interval t to $t + \Delta t$ the displacement would be

$$\Delta s = s(t + \Delta t) - s(t)$$

Example

Suppose the height of a canon ball above the ground is given by

$$s(t) = -5t^2 + 10t$$

What is the displacement between times $t = 1$ and $t = 4$?

Solution

$$\frac{\Delta s}{\Delta t} = \frac{s(t=4) - s(t=1)}{4-1}$$

$$= \frac{-5(4)^2 + 10(4) + 500(1)^2 - 10(1)}{3}$$

$$= \frac{-80 + 400 + 500 - 10}{3}$$

$$= \frac{810}{3} \qquad \text{(positive because } \Delta s > 0\text{)}$$

Example

When would the canon ball return to earth ?

Solution:

That would correspond to $s = 0$, so

$$0 = -5t^2 + 10t$$
$$= t(-5t + 10)$$

$$\Rightarrow t = 0 \quad \text{or} \quad t = 2$$

$t = 0$ is the time when the canon was fired, $t = 2$ is the time required for the canon ball to return to earth.

Def The <u>average velocity</u> of an object over a time interval $[t, t+\Delta t]$ is

$$\text{Average velocity} = \frac{\Delta s}{\Delta s}$$

$$= \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

The <u>instantaneous velocity</u> at time t is:

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

$$= s'(t)$$

where $s(t)$ is displacement as a function of time. Note, generally when we speak of velocity we mean instantaneous velocity.

Example

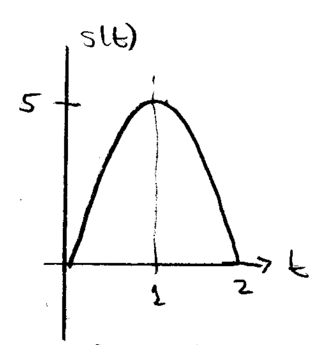
What is the velocity as a function of time of the canon ball in the example on the previous page?

Solution:

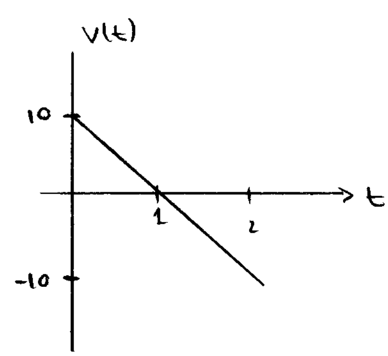
$$s(t) = -5t^2 + 10t$$

$$\Rightarrow v(t) = s'(t) = -10t + 10$$

Note the graphs:



$s \geq 0$ for $t \in [0,2]$



$v \geq 0$ for $t = [0,1]$, $v \leq 0$ for $t = [1,2]$

Def The <u>speed</u> of an object is:

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

Example

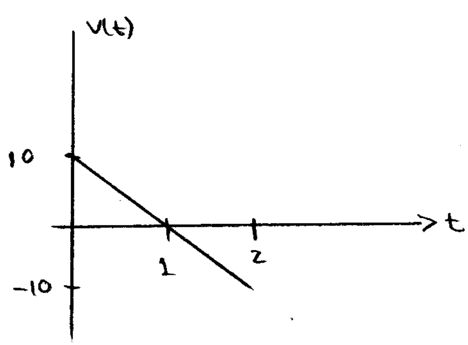
In the previous example;

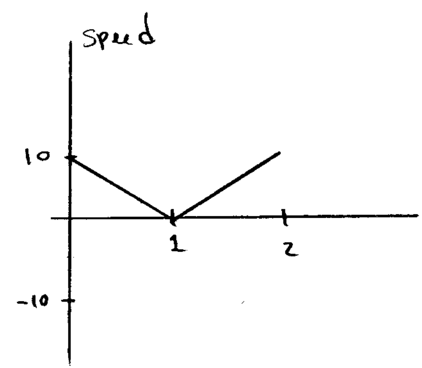
$$v(t) = -10t + 10 \qquad t \in [0,2]$$

and

$$\text{speed} = |v(t)|$$

$$= |-10t+10|$$





Def If $s(t)$ is the position of a body at time t, its <u>acceleration</u> is

$$a(t) = \frac{d^2}{dt^2}s(t)$$

$$= \frac{d}{dt}\left(\frac{ds}{dt}\right)$$

$$= \frac{d}{dt}v(t)$$

Example

(#11, p.140)

A rock thrown upward reaches a height $s(t) = 24t - 4.9t^2$ meters in t seconds.

a) Find the rock's velocity and acceleration at time t.

b) How long would it take the rock to reach its highest point?

c) How high would the rock go?

d) How long would it take the rock to reach half its maximum height.

e) How long would the rock be aloft?

Solution:

a)

$$s(t) = 24t - 4.9t^2 \quad \text{(meters)}$$

$$v(t) = s'(t) = 24 - 2(4.9)t$$

$$= 24 - 9.8t \quad \text{(meters/sec)}$$

$$a(t) = v'(t) = -9.8 \quad \text{(meter/sec}^2\text{)}$$

b) The velocity of the rock is initially positive, meaning $s(t)$ is increasing as the height of the rock increases. But as s increases, $v(t)$ decreases until it reaches zero. Thus s stops increasing. Subsequently the velocity becomes negative and s starts decreasing. The maximum height is thus obtained at the time when $v(t) = 0$. Thus:

$$0 = 24 - 9.8t$$

$$\Rightarrow t = \frac{24}{9.8}$$

$$\approx 2.45 \text{ s}$$

c) $s\left(\frac{24}{9.8}\right) = 24\left(\frac{24}{9.8}\right) - 4.9\left(\frac{24}{9.8}\right)^2$

≈ 29.4 m

d) Solve for time given $s(t) = \frac{1}{2}(29.4\ \text{m})$:

$$\frac{1}{2}(29.4) = 24t - 4.9t^2$$

$$\Rightarrow 4.9t^2 - 24t + 14.7 = 0$$

$$\Rightarrow t = \frac{24 \pm \sqrt{(24)^2 - 4(4.9)(14.7)}}{2(4.9)}$$

$$= \frac{24 \pm 16.9}{9.8}$$

$$\approx 0.7 \text{ or } 4.2 \text{ sec}$$

The half max height is achieved at ≈ 0.7 s on the way up and at 4.2 s on the way down.

e) The rock is aloft so long as $s > 0$, it is on the ground when $s = 0$. So solving for $s(t) = 0$:

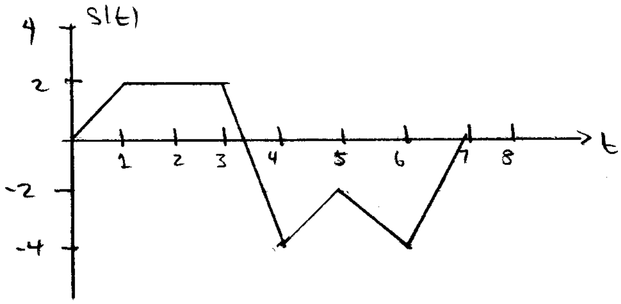
$$0 = 24t - 4.9t^2$$
$$= t(24 - 4.9t)$$

$$\Rightarrow t = 0 \quad \text{or} \quad t = 4.9\ \text{s}$$

The $t = 0$ is when the rock is initially thrown. It impacts 4.9 s later.

Example

Consider a particle moving forward and backway long a track. Let the particle's position relative to some coordinate system be given by



Assume a positive velocity means moving forward and a negative velocity means moving backward.

a) At what times is the particle moving forword? backword?

b) When is the particle at rest (ignore cases where the velocity is undefined)

Solution:

a) $v(t) = \frac{d}{dt} s(t)$ is the slope of the $s(t)$ graphed above. The slope is positive for $t \in [0,1]$, $t \in [4,5]$, and $t \in [6,7]$. The particle is moving forward during those times. It moves backword when $v(t) < 0$, this occurs for $t \in [3,4]$ and $t \in [5,6]$.

b) $v(t) = 0$ when the slope of $s(t)$ is zero. This occurs for $t \in [1,3]$. Note that $v(t)$ is undefined at $t = 1$, $t = 3$, $t = 4$, $t = 5$, and $t = 6$ because the function $s(t)$ is not differentiable at those points.

Free-Fall

If you drop an object in a vacuum, the distance it falls as a function of time is given by

$$\boxed{s(t) = \frac{1}{2} g t^2}$$

This is valid only near the Earth's surface (or near the surface of whatever planet, asteroid, etc... you are dropping your object on). The parameter g is a constant called the "acceleration due to gravity". This parameter is different for different gravitational bodies:

Location	g (m/s^2)
Sun	274.13
Mercury	3.59
Venus	8.87
Earth	9.81
Moon	1.62
Mars	3.77
Jupiter	25.95
Saturn	11.08
Uranus	10.67
Neptune	14.07
Pluto	0.42

So on Earth, $g = 9.8\ m/s^2$ (or $32.2\ ft/s^2$ if you absolutely insist on English units).

Given

$$s(t) = \frac{1}{2}gt^2$$

then the velocity is $\boxed{v(t) = gt}$

and the acceleration is

$$\boxed{a(t) = g}$$

(which explains why g is the "acceleration due to gravity").

<u>Example</u>

How many seconds does it take for an object released from rest to fall 980 m on Earth?

Solution:

$$980 = \frac{1}{2}(9.8)t^2$$

$$\Rightarrow t = \sqrt{\frac{2(980)}{9.8}}$$

$$= \sqrt{200}$$

$$= 10\sqrt{2} \text{ seconds}$$

Example

Consider a rock dropped from rest. How far will it have fallen at the point where the velocity is 100 m/s.

Solution:

Use $v = gt$ to determine the time at which the velocity is 100 m/s, then compute the distance fallen in that time:

$$t = \frac{v}{g}$$

$$= \frac{100 \text{ m/s}}{(9.8 \text{ m/s}^2)}$$

$$= 10.2 \text{ seconds}$$

$$\Rightarrow \quad s(10.2) = \frac{1}{2}(9.8)(10.2)^2$$

$$= 509.8 \text{ m}$$