APPM 1350: Section 2.6: Implicit Differentiation and Rational Exponents

Let's start with a result I discussed in passing on Friday:

> **The General Power Rule** If $a \in \mathbb{R}$ then x^a is differentiable at every interior point x of the domain of x^{a-1} and
>
> $$\frac{d}{dx} x^a = a x^{a-1}$$

The book introduces this for rational numbers a, but the result holds for any real exponent.

Example

$$\frac{d}{dx}(3+4x-x^2)^{1/2} = \underbrace{\tfrac{1}{2}(3+4x-x^2)^{-1/2}}_{\text{outer}} \; \underbrace{(4-2x)}_{\text{inner}}$$

$$= \frac{2-x}{\sqrt{3+4x-x^2}}$$

where we have used both the chain rule and the power rule $\left(\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}\right)$.

Example

(#8, p. 170)

$$\frac{d}{dx}(1-6x)^{2/3} = \underbrace{\tfrac{2}{3}(1-6x)^{-1/3}}_{\text{outer}} \; \underbrace{(-6)}_{\text{inner}}$$

$$= -\frac{4}{(1-6x)^{1/3}}$$

Implicit Differentiation

A function

$$y = f(x)$$

associates a unique value y for every value of x. By the same token this equation implicitly defines x as a function of y. More generally, consider a function like

$$xy + x - 2y - 1 = 0 \qquad (1)$$

This has the form

$$f(x, y) = 0$$

Here the x's and y's are mixed together and we can say that the formula implicitly defines y as a function of x or visa versa. To make an explicit formula for y as a function of x we would do the usual factoring:

$$xy + x - 2y - 1 = 0$$

$$\Rightarrow y(x-2) + x - 1 = 0$$

$$\Rightarrow y = \frac{1-x}{x-2} \qquad (2)$$

In this particular case we can also come up with an explicit formula for x as a function of y:

$$xy + x - 2y - 1 = 0$$

$$\Rightarrow x(y+1) - 2y - 1 = 0$$

$$\Rightarrow x = \frac{2y+1}{y+1} \qquad (3)$$

The important point is that we can think of (1) as expressing information about y as a function of x or as expressing information about x as a function of y. So what can we say about the derivatives of these functions?

Go back to (1) and treat y as a <u>function</u> of x. Then

$$\frac{d}{dx}(xy + x - 2y - 1) = \frac{d}{dx}0$$

$$\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}x - 2\frac{d}{dx}y = 0$$

$$\Rightarrow \underbrace{x\frac{dy}{dx} + y\frac{dx}{dx}}_{\substack{\text{product rule} \\ \text{on } \frac{d}{dx}xy}} + \frac{dx}{dx} - 2\frac{dy}{dx} = 0$$

$$\Rightarrow x\frac{dy}{dx} + y + 1 - 2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x-2) + y + 1 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+y}{2-x} \qquad (4)$$

We have derived an equation for dy/dx, but it is defined in terms of both x <u>and</u> y. This is called <u>implicit differentiation</u> and in this case we have defined the derivative of y with respect to x <u>implicitly</u> in terms of a function of both x and y. Given a point (x, y) that satisfies the original equation

$$xy + x - 2y - 1 = 0$$

this implicit derivative can be evaluated to tell us the derivative of y as a function of x at that point.

So is (4) <u>really</u> the derivative of the explicit formula in (2)? Differentiate (2) and find out:

$$y = \frac{1-x}{x-2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x-2)\frac{d}{dx}(1-x) - (1-x)\frac{d}{dx}(x-2)}{(x-2)^2}$$

$$= \frac{-(x-2) - (1-x)}{(x-2)^2}$$

$$= \frac{-\frac{(x-2)}{(x-2)} - \frac{(1-x)}{(x-2)}}{(x-2)}$$

$$= \frac{1 + \frac{1-x}{x-2}}{2-x}$$

$$= \frac{1+y}{2-x}$$

which is what we found by implicit differentiation on the previous page. So yes, it works! And it was a lot easier to do the implicit differentiation (in this case) than solving for y in terms of x and then doing the explicit differentiation above.

<u>Example</u>

Find $\frac{dy}{dx}$ given $x^2y - xy^2 + x^2 + y^2 = 0$

Solution:

$$\frac{d}{dx}(x^2y - xy^2 + x^2 + y^2) = \frac{d}{dx}0$$

$$\Rightarrow x^2\frac{dy}{dx} + y\frac{d}{dx}x^2 - x\frac{d}{dx}y^2 - y^2\frac{d}{dx}x + \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

$$\Rightarrow x^2\frac{dy}{dx} + 2xy - x\,2y\frac{dy}{dx} - y^2 + 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}\left(x^2 - 2xy + 2y\right) = -2xy + y^2 - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - 2xy - 2x}{x^2 - 2xy + 2y}$$

<u>Example</u>

The total surface area of a right circular cylinder of radius r and height h is given by $S = 2\pi r^2 + 2\pi r h$. If S is a constant, find dr/dh.

Solution:

$$\frac{d}{dh} S = \frac{d}{dh}\left(2\pi r^2 + 2\pi r h\right)$$

$$\Rightarrow 0 = 2\pi \frac{d}{dh} r^2 + 2\pi r \frac{d}{dh} h + 2\pi h \frac{d}{dh} r$$

$$\Rightarrow 0 = 2\pi \left(2r \frac{dr}{dh}\right) + 2\pi r + 2\pi h \frac{dr}{dh}$$

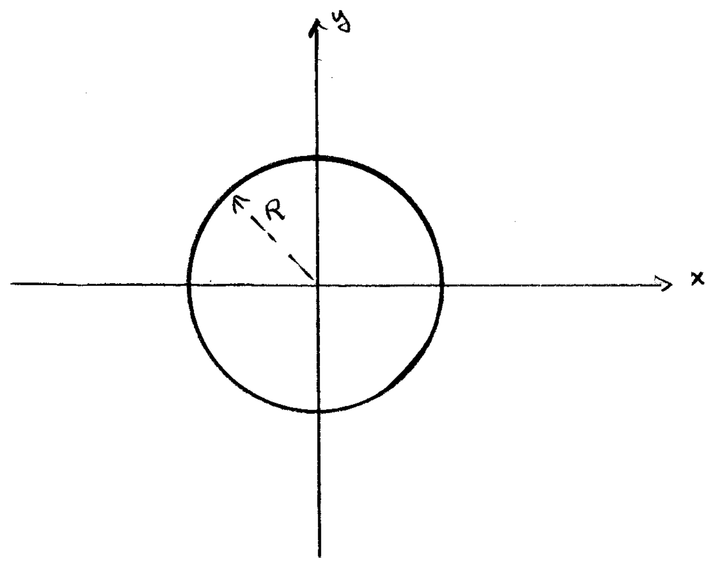
$$\Rightarrow 0 = \frac{dr}{dh}\left(4\pi r + 2\pi h\right) + 2\pi r$$

$$\Rightarrow \frac{dr}{dh} = -\frac{2\pi r}{4\pi r + 2\pi h}$$

$$= -\frac{r}{2r + h}$$

Implicit Differentiation on General Space Curves

In the examples above we had well behaved functions with one value of y for every value of x. But suppose we have something like a circle of radius R:



The equation of this circle is

$$x^2 + y^2 = R^2$$

This equation implicitly associates values of x with values of y. But note that there are two values of y for every x. So it is not a true function in the usual sense. Rather, the circle is the union of the two functions:

$$y(x) = \sqrt{R^2 - x^2} \qquad (y \geq 0) \qquad (5)$$

and

$$y(x) = -\sqrt{R^2 - x^2} \qquad (y \leq 0) \qquad (6)$$

The derivative $\frac{dy}{dx}$ of either of these functions is well defined and we can easily compute them from 15) or 16). However, it turns out we can more easily compute the derivative implicity, Specifically,

$$x^2 + y^2 = R^2 \qquad (R = \text{constant})$$

$$\Rightarrow \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}R^2$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

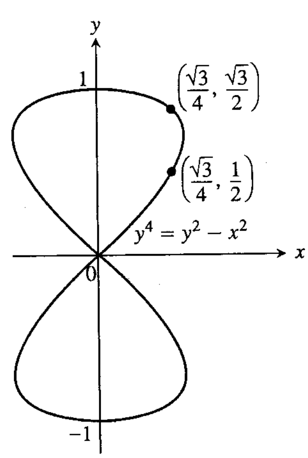
$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y} \qquad (y \neq 0)$$

This works for the entire circle $(y \neq 0)$, we now don't need to consider the upper semicircle versus the lower semi-circle. At $y=0$ the slope is undefined, which makes sense in that the tangent line at $y=0$ is vertical (this happens when $x = \pm R$).

Example

(#59, p. 170)

59. *The eight curve.* Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



Solution:

$$\frac{d}{dx} y^4 = \frac{d}{dx}(y^2 - x^2)$$

$$4y^3 \frac{dy}{dx} = 2y\frac{dy}{dx} - 2x$$

$$\Rightarrow \frac{dy}{dx}(4y^3 - 2y) = -2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{2y - 4y^3}$$

$$= \frac{x}{y - 2y^3}$$

At $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)$;

$$\frac{dy}{dx} = \frac{\left(\frac{\sqrt{3}}{4}\right)}{\frac{1}{2} - 2\left(\frac{1}{2}\right)^3}$$

$$= \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - \frac{1}{4}}$$

$$= \sqrt{3}$$

At $\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$;

$$\frac{dy}{dx} = \frac{\left(\frac{\sqrt{3}}{4}\right)}{\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)^3}$$

$$= \frac{(\sqrt{3}/4)}{\sqrt{\frac{3}{2}}\left(1 - \frac{3}{2}\right)}$$

$$= -\frac{\sqrt{2}}{2}$$

Implicit Differentiation and Higher Order Derivatives

Higher order derivatives can be taken implicitly by differentiating multiple times.

Example

Find y'' given $x + xy + y = 2$

Solution:

$$\frac{d}{dx}(x + xy + y) = \frac{d}{dx} 2$$

$$\Rightarrow \frac{dx}{dx} + x\frac{dy}{dx} + y\frac{dx}{dx} + \frac{dy}{dx} = 0$$

$$\Rightarrow 1 + x\frac{dy}{dx} + y + \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x+1) + (y+1) = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(1+y)}{(1+x)}$$

Next:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} - \frac{(1+y)}{(1+x)}$$

$$= -\frac{(1+x)\frac{d}{dx}(1+y) - (1+y)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= -\frac{(1+x)\frac{dy}{dx} - (1+y)}{(1+x)^2}$$

$$= \frac{\cancel{(1+x)}\left(\frac{1+y}{\cancel{1+x}}\right) + (1+y)}{(1+x)^2}$$

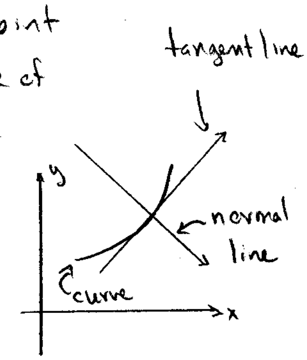
substitute $-\frac{1+y}{1+x}$ for $\frac{dy}{dx}$ in numerator.

$$= \frac{2(1+y)}{(1+x)^2}$$

Normals to Curves

Def A line is normal to a curve at a point if it is perpendicular to the curve's tangent there. The line is called the normal to the curve at that point.

We can find the equation of the normal to a curve in much the same fashion as we find tangent lines. Specifically, take the derivative at a point to find the slope m of the tangent line at (x_0, y_0). The slope of the normal is the negative reciprocal. Hence the equation of the tangent line at (x_0, y_0) with slope m is



$$y_t(x) = y_0 + m(x - x_0)$$

and the equation of the normal line is:

$$y_n(x) = y_0 - \frac{1}{m}(x - x_0) \qquad (m \neq 0)$$

Example

Find the equations of the tangent and normal lines to $y = x^3 - 2x^2 + 4$ at $(2,4)$.

Solution:

$$\frac{dy}{dx} = 3x^2 - 4x$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(2,4)} = 3(2)^2 - 4(2) = 4$$

Hence:

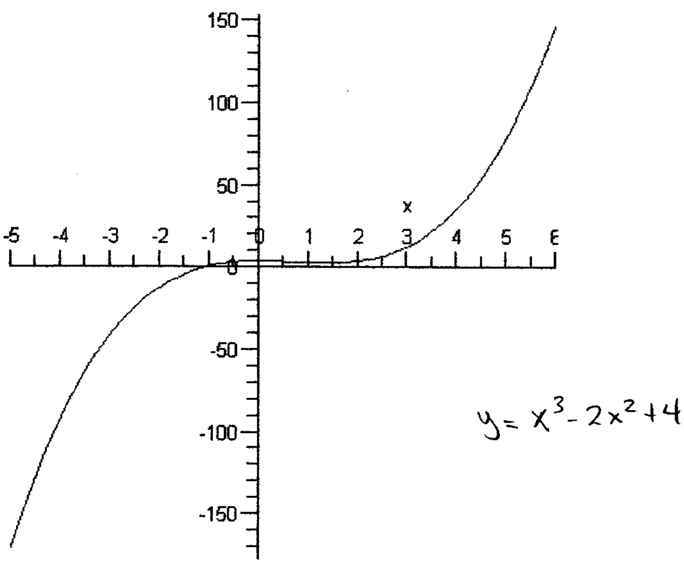
$$y_t(x) = 4 + 4(x-2)$$

$$= 4x - 4$$

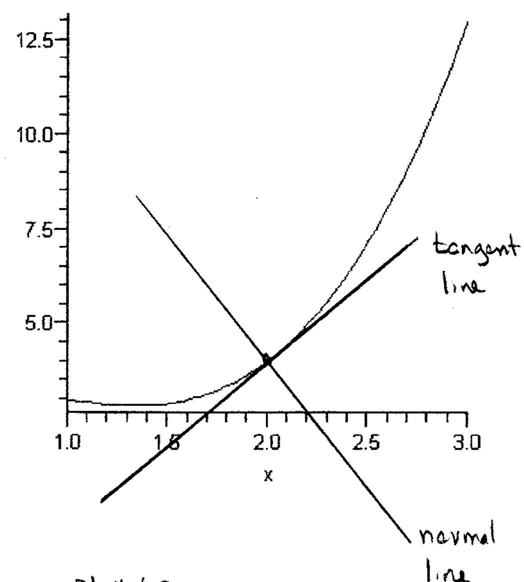
$$y_n(x) = 4 - \frac{1}{4}(x-2)$$

$$= \frac{1}{4}x + \frac{9}{2}$$

← slope of normal is the negative reciprocal of the slope of the tangent line.



Plotted for $-5 \leq x \leq 6$



Plotted for $1 \leq x \leq 3$

Example

Find equations for the tangent and normal lines to $x^2 - y^2 = 7$ at the point $(4, -3)$.

Solution:

We need $\frac{dy}{dx}$, compute it implicitly.

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx} 7$$

$$\Rightarrow \quad 2x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}$$

Hence the slope at the tangent line at $(4, -3)$ is

$$m = \left.\frac{dy}{dx}\right|_{(4,-3)}$$

$$= \frac{4}{-3}$$

$$= -\frac{4}{3}$$

Thus the equation of the tangent line is:

$$y = -3 - \frac{4}{3}(x - 4)$$

$$\Rightarrow \quad 3y = -9 - 4(x - 4)$$

$$\Rightarrow \quad 4x + 3y = 7$$

and for the normal line:

$$y = -3 + \frac{3}{4}(x-4)$$

$$\Rightarrow \quad 4y = -12 + 3x - 12$$

$$\Rightarrow \quad 3x - 4y = 24$$