APPM 1350: Section 3.7: Linearization and Differentials

Linearizations

Consider a function $f(x)$ that is differentiable in an open interval around a point a. By definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \frac{f(x) - f(a)}{x - a} + \tilde{\epsilon}(x) \qquad \tilde{\epsilon} = \text{some function}$$

So in place of the limit we use the difference quotient and then add in a function $\tilde{\epsilon}(x)$ to make up for not taking the limit. Rearranging this equation we have:

$$f'(a)(x-a) = f(x) - f(a) + \tilde{\epsilon}(x)(x-a)$$

$$\Rightarrow f(x) = f(a) + f'(a)(x-a) + \underbrace{\epsilon(x)(x-a)}_{\text{error}}$$

where $\epsilon(x) = -\tilde{\epsilon}(x)$. We know that as $x \to a$, that $\epsilon(x) \to 0$ since in the limit we recover the derivative of f at a. So for x close to a (we write this as $|x-a| << 1$) we get a pretty good approximation to the derivative by dropping the error term and just writing

$$f(x) \approx f(a) + f'(a)(x-a)$$

This says that $f(x)$ can be approximated near a by a linear function and is hence called a <u>linearization</u>.

Formally:

Def If f is differentiable at $x=a$, then the approximating function

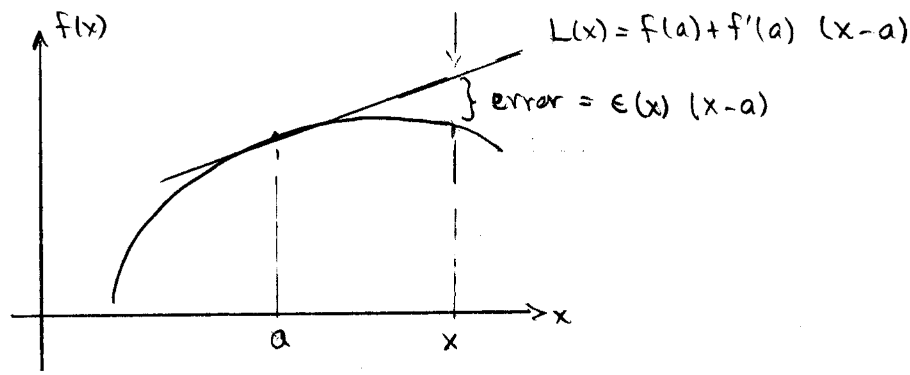
$$L(x) = f(a) + f'(a)\,(x-a)$$

is the <u>linearization</u> of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the <u>standard linear approximation</u> of f at a. The point $x=a$ is the <u>center</u> of the approximation.

Graphically, the linearization amounts to approximating the function by its tangent line and the linear approximation is simply the equation of the tangent line at $x=a$.



<u>Example</u>

Find the linearization of $f(x) = \sqrt{x}$ at $x=1$

Solution:

$$f(x) = \sqrt{x} = x^{1/2}$$

$$\Rightarrow \quad f'(x) = \tfrac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad f(x) \approx L(x) = f(1) + f'(1)\,(x-1)$$

$$= 1 + \tfrac{1}{2}(x-1)$$

$$= \tfrac{1}{2}(1+x)$$

How good is the approximation?

x	$\sqrt{x}$	$\frac{1}{2}(x+1)$	Error $(f(x)-L(x))$
0	0	0.50	-0.50
0.5	0.707	0.75	-0.043
0.7	0.837	0.85	-0.013
0.8	0.894	0.90	-0.006
0.9	0.949	0.95	-0.001
1.0	1.0	1.0	0
1.1	1.048	1.05	-0.002
1.2	1.095	1.1	-0.005
1.3	1.140	1.15	-0.010
1.5	1.225	1.25	-0.025
2.0	1.414	1.5	-0.086

So this isn't a bad approximation near $x=1$. But watch out, at $x=9$:

$$f(9)=\sqrt{9}=3 \qquad L(3)=\tfrac{1}{2}(9+1)=5$$

So the error is now equal to 2. That is not such a good approximation.

<u>Example</u>

Find the linearization of $f(x)=\sin x$ at $x=0$.

Solution:

$$f(x)=\sin x$$
$$f'(x)=\cos x$$

$$\Rightarrow L(x) = f(0)+f'(0)(x-0)$$
$$= 0 + 1(x)$$
$$= x$$

Example

Find the linearization of $f(x) = (1+x)^k$ where $k \in \mathbb{R}$ at $x=0$.

Solution:

$$f(x) = (1+x)^k$$
$$f'(x) = k(1+x)^{k-1} \qquad \text{(chain \& power rules)}$$

$$\Rightarrow \quad L(x) = f(0) + f'(0)(x-0)$$
$$= 1^k + k(1)^{k-1}(x-0)$$
$$= 1 + kx$$

So near $x=0$, $\quad (1+x)^k \approx 1 + kx$

Differentials

Return to our linearization of $f(x)$ at $x=a$:

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$\Rightarrow \underbrace{f(x) - f(a)}_{\Delta f} \approx f'(a) \underbrace{(x-a)}_{\Delta x}$$

$$\Rightarrow \Delta f \approx f'(a)\,\Delta x$$

Δf is the error in the linear approximation to f at a if we move a distance Δx. But as we have noted, if we let $|\Delta x| << 1$, then this result becomes more and more exact. For very small Δx we replace Δx by the symbol dx and replace Δf by the symbol df and write

$$df = f'(x)\,dx$$

at some point x. The dx and df are called <u>differentials</u>. Note that I wrote an equality here rather than an approximation. We are actually defining df to be equal to $f'(x)\,dx$ at x for some value dx.

Formally:

> **Def** Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable (similiar to Δx). The differential df is defined by
>
> $$df = f'(x)\,dx$$

What $df = f'(x)\,dx$ means is that if we move a small distance dx from x, then the change in $f(x)$ near x is approximately equal to df. As dx gets smaller and smaller the approximation gets better and better. Put only very slightly differently:

> Let $f(x)$ be differentiable at x. The approximate change in the value of f when x changes from x to $x + dx$ is
>
> $$df = f'(x)\,dx$$

Note that df is a function of x.

Example

(# 21, p. 258) Given $f(x) = x^3 - 3\sqrt{x}$, find df

Solution:

$$f(x) = x^3 - 3\sqrt{x}$$
$$f'(x) = 3x^2 - \tfrac{3}{2}x^{-1/2}$$

$$\Rightarrow \quad df = f'(x)\,dx$$
$$= \left(3x^2 - \frac{3}{2\sqrt{x}}\right)dx$$

Note what this means. If we move a distance dx from x, then the change in the tangent line to f(x) at x is df. And the change in f(x) itself in moving from x to x+dx is approximately df. So; let x= 1 and let dx= 0.1. Then

$$
\begin{aligned}
df &= f'(1)\,(0.1) \\
&= \left(3 - \tfrac{3}{2}\right)(0.1) \\
&= 0.150
\end{aligned}
$$

How good an approximation is this to the real change in f(x)? The real change is:

$$
\begin{aligned}
\Delta f &= f(x+dx) - f(x) \\
&= f(1+0.1) - f(1) \\
&= \underbrace{(1.1)^3 - 3\sqrt{1.1}}_{f(x+dx)} - \underbrace{\left[1^3 - 3\sqrt{1}\right]}_{f(x)} \\
&= -1.8154 - (-2) \\
&= 0.184
\end{aligned}
$$

$\Delta f = f(x+dx) - f(x)$ is called the <u>absolute change</u> in f(x) going from x to dx. and is the true change. The <u>approximation to the absolute error</u> is df = 0.15. The difference between the true absolute change and the approximate absolute change is called the <u>approximation error</u>:

$$\boxed{\text{Approximation error} = \Delta f - df}$$

In our example;

$$\text{Approximation error} = 0.184 - 0.150 = 0.034.$$

There are other kinds of change, both true and approximate:

Def:

Type of change	True	Approximate
Absolute change	$\Delta f = f(x+dx) - f(x)$	df
Relative change	$\frac{\Delta f}{f(x)}$	$\frac{df}{f(x)}$ $(f(x) \neq 0)$
Percentage change	$\frac{\Delta f}{f(x)} \times 100$	$\frac{df}{f(x)} \times 100$ $(f(x) \neq 0)$

Again, using our example, we have (note that $f(1) = \frac{3}{2} = 1.5$):

True absolute change $= 0.184$

Approx " " $= 0.150$

True relative change $= \frac{0.184}{1.5} = 0.1227$

Approx relative change $= \frac{0.150}{1.5} = 0.1$

True percentage change $= \frac{0.184}{1.5} \times 100 = 12.27\%$

Approx percentage change $= \frac{0.150}{1.5} \times 100 = 10\%$

And returning to approximation error, we had

$$\begin{aligned} \text{approximation error} &= \Delta f - df \\ &= \Delta f - f'(x)\,dx \end{aligned}$$

We define the error to equal the value $\epsilon(x)\,dx$. Then

$$\text{approx. error} = \epsilon(x)\,dx = \Delta f - f'(x)\,dx$$

$$\Rightarrow \Delta f = f'(x) + \epsilon(x)\,dx$$

which is actually where we started on page 1. Formally;

> If $y = f(x)$ is differentiable at $x = x_0$ and x changes from x_0 to $x_0 + dx$, the change Δf in f is given by
>
> $$\Delta f = f'(x_0)\,dx + \epsilon(x)\,dx$$
>
> in which $\epsilon \to 0$ as $dx \to 0$.

<u>Example</u>

(# 40, p. 258) Estimate the change in the volume $V = x^3$ of a cube when the edge lengths change from x_0 + $x_0 + dx$

Solution:

$$V = x^3$$

$$\Rightarrow V'(x) = 3x^2$$

$$\Rightarrow df = V'(x)\,dx$$

$$= 3x^2\,dx$$

So at $x = x_0$; the approximate absolute change is:

$$df(x_0) = 3x_0^2\,dx$$

Example

(#46, p. 259) The diameter of a tree was 10 inches. During the following year the circumference grew 2 inches. Approximately how much did the tree's diameter and cross-section area change?

Solution:

Assume the cross-section of the tree is a circle of diameter D. The circumference is:

$$C = \pi D$$

$$\Rightarrow dC = \pi \, dD$$

$$\Rightarrow dD = \frac{dC}{\pi}$$

Since $dC = 2$ inches, we have that the diameter approximately changed by

$$dD = \frac{dC}{\pi}$$

$$= \frac{2}{\pi} \text{ inches}$$

As for the cross-section A,

$$A = \pi \left(\frac{D}{2}\right)^2$$

$$= \frac{\pi}{4} D^2$$

$$\Rightarrow dA = \frac{\pi}{4}(2D)\, dD$$

$$\Rightarrow dA = \frac{\pi}{2} D \, dD$$

$$= \frac{\pi}{2}(10)(2) \text{ in}^2$$

$$= 10\pi \text{ in}^2$$

Example

(#48, p. 259) About how accurately should you measure the side of a square to be sure of calculating the area within 2% of its true value?

Solution:

Let the length of the edges be l. The area is then

$$A = l^2$$

$$\Rightarrow dA = 2l\,dl$$

$$\Rightarrow \underbrace{\frac{dA}{A}100}_{\substack{\text{approx}\\ \text{percentage}\\ \text{change}\\ = 2\%}} = \frac{2l}{A}100\,dl$$

$$\Rightarrow 2 = \frac{2l}{A}100\,dl$$

$$\Rightarrow dl = \frac{A}{100l}$$

$$= \frac{l^2}{100l}$$

$$= \frac{l}{100}$$

So we need to measure the edge to within a length $\frac{l}{100}$. Or in percentage terms:

$$\frac{dl}{l} \times 100 = \frac{l}{100}\ \frac{100}{l}$$

$$= 1\%$$

The Relation Between Differentials and Derivatives

When we write

$$df = f'(x)\,dx \qquad (1)$$

we assume $dx \neq 0$. So we can divide both sides by dx to get:

$$\frac{df}{dx} = f'(x) \qquad (2)$$

So the ratio of df to dx is just the derivative (this is a bit tautological, this works because this is how we defined df). Consequently these are equivalent ways of writing a derivative. So if you see equation (1), recognize that it is equivalent to writing equation 2 and visa versa. By the same token and for the same reason differentials obey the same rules as derivatives:

$$\boxed{\begin{aligned}
&dc = 0 && (c = \text{constant})\\
&d(cu) = c\,du && (c = \text{constant})\\
&d(u+v) = du + dv\\
&d(uv) = v\,du + u\,dv\\
&d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}\\
&d(u^n) = n u^{n-1}\,du\\
&d(\sin u) = \cos u\,du\\
&d(\cos u) = -\sin u\,du\\
&d(\tan u) = \sec^2 u\,du\\
&d(\cot u) = -\csc^2 u\,du\\
&d(\sec u) = \sec u \tan u\,du\\
&d(\csc u) = -\csc u \cot u\,du
\end{aligned}}$$