APPM 1350: Section 3.8: Newton's Method

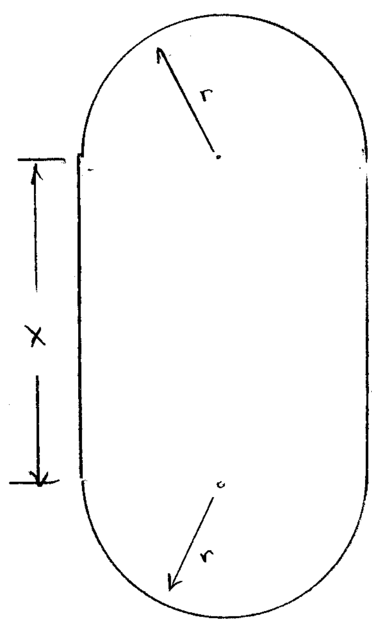
Review

Class exercise

(# 68, p. 271)

An athletic field is to be built in the shape of a rectangle x units long capped by semicircular regions of radius r at the two ends. The field is to be bounded by a 400 m racetrack. What values of x and r give the rectangle the largest possible area?

Solution:



The circumference of the field is 400m. Hence:

$$C = 2x + 2\pi r = 400 \quad (1)$$

The area of the rectangle is

$$A = xr \quad (2)$$

From (1): $400 = 2x + 2\pi r$

$$\Rightarrow r = \frac{400 - x}{\pi}$$

$$\Rightarrow A = xr$$

$$= \frac{(400 - x)x}{\pi}$$

We want A to be maximized. Thus

$$\frac{dA}{dx} = 0 = \frac{1}{\pi}(400 - 2x)$$

$$\Rightarrow \quad x = 200 \text{ m}$$

Check that this is a maximum:

$$\frac{d^2A}{dx^2} = -\frac{2}{\pi} < 0$$

so it is indeed a maximum.

An interesting example

Lay a rope around the equator of the Earth. Its length will be approximately 25,000 miles. Next raise the rope ten feet at all points. How much longer a rope is required?

Solution:

Assume the Earth is a sphere, in which case the circumference is:

$$C(r) = 2\pi r$$

We are interested dC when $dr = 10$ feet and $C = 2500$ miles. Hence

$$\begin{aligned} dC &= 2\pi\, dr \\ &= 2\pi\,(10 \text{ ft}) \\ &= 20\pi \text{ ft} \\ &= 0.004\pi \text{ mi} \end{aligned}$$

Note that dC is independent of r! (at least for small dr).

The percentage change is:

$$\frac{dC}{C} \times 100 = \frac{0.004\pi}{2500} \times 100 = 4.76 \times 10^{-4} \%$$

Finding Roots with the Bisection Method

Suppose we have a function $f(x)$ and we want to find the value of x where $f(x)=0$ on some interval $x=[a,b]$. First, is there a zero in that interval? If $f(x)$ is continuous and $f(a)$ and $f(b)$ have opposite signs then by the intermediate value theorem there will indeed be some $x_0 \in (a,b)$ where $f(x_0)=0$; i.e, where $f(x)$ has a **root.** Then a first guess for x_0 would be

$$x_0 = \frac{a+b}{2}$$

i.e. split the difference between a and b. Now, if $f(x_0)=0$ (or close enough to satisfy us) then we are done. But suppose instead $f(x_0) \neq 0$ and the sign of $f(a)$ and $f(x_0)$ are the same. A root could lie between a and x_0, but we are guaranteed by the intermediate value theorem that there is at least one root between x_0 and b because $f(x_0)$ and $f(b)$ have opposite signs. So go after that root by estimating the root is at

$$x_1 = \frac{x_0+b}{2}.$$

i.e, split the difference again between points where the function has opposite signs. Now repeat the logic again to figure out whether a root is on $[x_0, x_1]$ or $[x_1, b]$ (assuming $f(x_1) \neq 0$). Then bisect the interval again and just keep repeating the process until you get a point x_n where $|f(x_n)| < \epsilon$ and ϵ is whatever small number is close enough to zero to satisfy you that you have a good estimate of the root of f. This iterative process is called the <u>bisection method</u>.

Pseudocode for the bisection method:

```
loop
  m = (a+b) / 2
  if f(a) · f(m) < 0 then
     b = m
  else
     a = m
until |f(m)| < required accuracy
```

Note that often one uses

$$|b-a| < \text{required accuracy}$$

instead as a stepping criteria.

The bisection method is an algebraic technique that is perfectly general. However, it is slow (meaning it requires a lot of iterations to achieve any significant accuracy).

Example

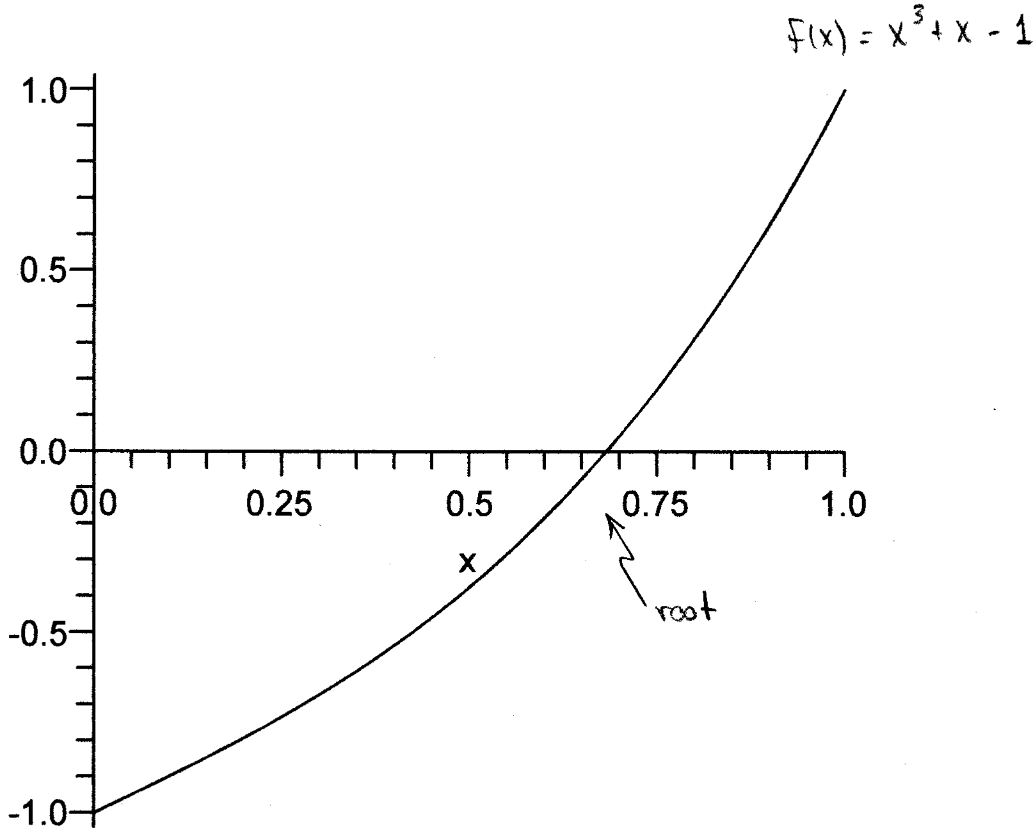
Estimate a root of $x^3 + x - 1$ on $x \in [0, 1]$ using the bisection method:

Iteration #	a	b	f(a)	f(b)	m	f(m)	|a-b|
1	0	1	-1	1	0.5	-0.375	1
2	0.5	1	-0.375	1	0.75	0.172	0.5
3	0.5	0.75	-0.375	0.172	0.625	-0.131	0.25
4	0.625	0.75	-0.131	0.172	0.6875	0.01245	0.125

If we stop when $|f(m)| \leq 0.0125$ then our root lies in $[0.625, 0.6875]$. The convergence is slow, but the algorithm is guaranteed to always converge (i.e., always find at least one root).

(The function is plotted on the next page)

$plot(x^3+x-1, x=0..1);$



Newton's Method

The bisection method has guaranteed convergence. However, it is a very inefficient numerical technique. Faster methods for finding roots can be derived using information from the derivative. However, all derivative based root finding techniques must be used with caution, they are very powerful but they are fraught with possible disaster.

The simplest derivative root finding technique is Newton's method or the Newton-Raphson method (its discovery is usually credited to Newton). Note from the difference quotient:

$$f'(x_0) \approx \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow f'(x_0)\,(x - x_0) \approx f(x) - f(x_0)$$

$$\Rightarrow f(x) \approx f(x_0) + f'(x_0)\,(x - x_0)$$

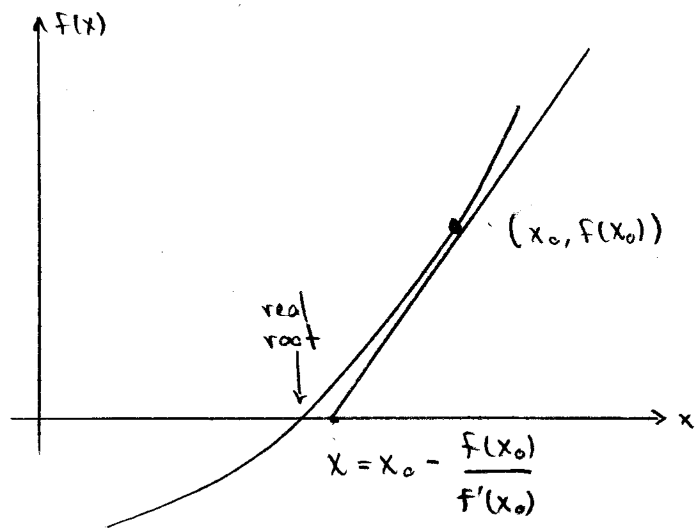
This should be looking familiar by now. The right-hand side is just the equation of the line tangent to $f(x)$ at x_0. The equation just says that when $|x - x_0|$ is small then $f(x)$ can be approximated by its tangent line (this is the linearization from section 3.7). Now suppose we are interested in finding x when $f(x) = 0$. Then lets set $f(x) = 0$ in our equation and solve for x.

$$0 \approx f(x_0) + f'(x_0)(x - x_0)$$

$$\Rightarrow x - x_0 \approx -\frac{f(x_0)}{f'(x_0)} \qquad (f'(x_0) \neq 0)$$

$$\Rightarrow x \approx x_0 - \frac{f(x)}{f'(x_0)}$$

This gives an estimate of x where $f(x) = 0$. In reality this is the point where the linearization $L(x) = f(x_0) + f'(x_0)(x - x_0)$ of $f(x)$ at x_0 is equal to zero. But if the linearization is accurate near the point where $f(x) = 0$, then this should be a decent estimate of the point where the root is.



Generally though we won't get that great a result on the first try. So we let x_0 be the root x we found above and repeat the process until we get the accuracy we want.

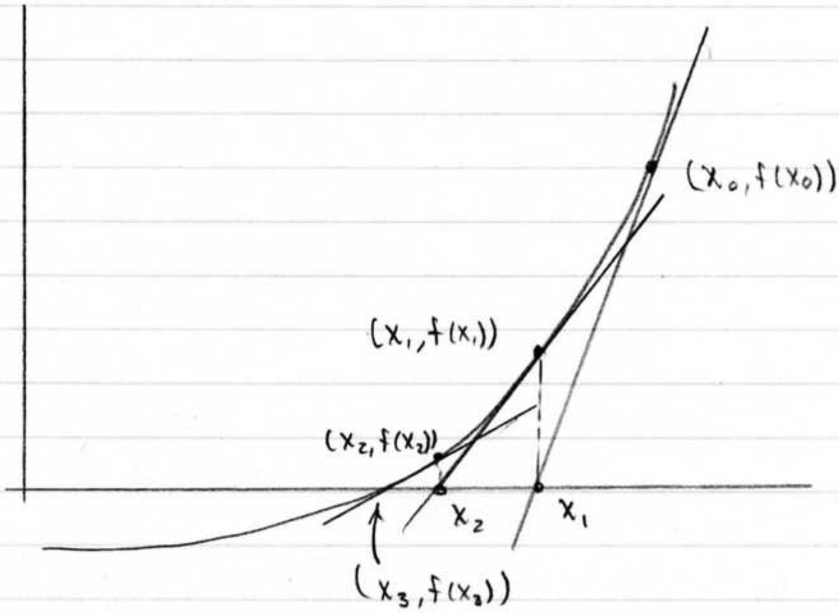
<u>Psuedocode for Newton's Method</u>

```
x = x0      (some initial guess at the root)
loop
   x = x - f(x) / f'(x)
until |f(x)| < required accuracy
```

In short, repeatedly replace your estimate of the root x with

$$\boxed{x_{next} = x - \frac{f(x)}{f'(x)}}$$

provided $f'(x) \neq 0$.



So long as Newton's law works the rule of thumb is that each iteration doubles the places of accuracy.

Example

Let's return to the same function on page 4 that we solved with the bisection method, i.e., $x^3 + x - 1$ on $[0, 1]$, but this time find the root using Newton's method. As a starting point use the bisection of the interval (i.e., $x_0 = \frac{a+b}{2}$ if we are on $[a, b]$) since we already know that there is a root between $x = 0$ and $x = 1$ (from the intermediate value theorem). So

Iteration n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	$f(x_{n+1})$
1	0.5	-0.375	1.75	0.7143	0.079
2	0.7143	0.079	2.53	0.6832	0.002
3	0.6832	0.002	2.40	0.6823	0.000001

We reach the criteria $|f(x)| < 0.0125$ in three iterations rather than the four for bisection. But wow, look at the value of $f(x)$ when we converge. For bisection $f(x_4) = 0.0125$ but for Newton $f(x_3) = 0.000001$! It converged faster and a lot more accurately. Way to go Newton!

Example

Let's invent an algorithm for taking square roots by hand. Our goal is to find a positive root of

$$f(x) = x^2 - a = 0 \qquad a > 0$$

The Newton recursion will be:

$$x_{next} = x - \frac{f(x)}{f'(x)}$$

$$= x - \frac{x^2 - a}{2x}$$

$$= \frac{2x^2 - x^2 + a}{2x}$$

$$= \frac{x^2 + a}{2x}$$

$$= \frac{x + a/x}{2}$$

we can also write this as:

$$x_{n+1} = \frac{x_n + a/x_n}{2}$$

where the subscript indicates the iteration number. So, for example, let $a = 2$. Then we want $x^2 - 2 = 0 \Rightarrow x = \sqrt{2} = 1.414213562\cdots$. On the first iteration try $x_0 = 3/2$, then

$$x_1 = \frac{3/2 + 2/(3/2)}{2}$$

$$= \frac{17}{12}$$

$$= 1.41666\ldots$$

We got 3 digits of accuracy on the first iteration! Let's do one more

$$x_2 = \frac{\frac{17}{2} + \frac{2}{(17/2)}}{2}$$

$$= 1.414215$$

which is good to five places!

However... Newton's Method Does Not Always Converge!

Example

Try and find $f(x)=0$ for $f = x^{1/3}$ using Newton's method.

Solution:

We have

$$x_{next} = x - \frac{f(x)}{f'(x)}$$

$$= x - \frac{x^{1/3}}{(\frac{1}{3}x^{-2/3})}$$

$$= x - 3x$$

$$= -2x$$

The answer should be $x=0$ $\quad (f(x) = x^{1/3} = 0 \Rightarrow x = 0)$. But if we start Newton with some $x \neq 0$ note that rather than x getting closer to x it actually gets twice as far away on each iteration. In this case Newton's method fails to converge (i.e, it is divergent).

In this particular case this shouldn't be too surprising, $f'(0)$ does not even exist and the algorithm just gets confused near the origin.

So watch out, <u>Newton's method is powerful but it is not infallible.</u>

Another Example

(#5, p. 266)

Use Newton's method to find the positive fourth root of 2 by solving $x^4 - 2 = 0$. Start with $x_0 = -1$ and find x_2.

Solution:

$$f(x) = x^4 - 2$$
$$f'(x) = 4x^3$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^4 - 2}{4x_n^3}$$

$$= \frac{4x_n^4 - x_n^4 + 2}{4x_n^3}$$

$$= \frac{3x_n^4 + 2}{4x_n^3}$$

$$= \frac{3}{4}x_n + \frac{1}{2x_n^3}$$

So if $x_0 = -1$, then

$$x_1 = \frac{3}{4}x_0 + \frac{1}{2x_0^3}$$

$$= -\frac{3}{4} - \frac{1}{2}$$

$$= -\frac{5}{4}$$

$$f(x_1) = \left(-\frac{5}{4}\right)^4 - 2$$

$$= 0.441$$

$$x_2 = \frac{3}{4}x_1 + \frac{1}{2x_1^3}$$

$$= \frac{3}{4}\left(-\frac{5}{4}\right) - \frac{1}{2\left(\frac{5}{4}\right)^3}$$

$$= -\frac{15}{16} - \frac{32}{125}$$

$$= -1.1935$$

$$f(x_2) = 0.02903$$