APPM 1350: Section 4.1 : Indefinite Integrals

Derivatives are one of the greatest discoveries of all times. We now turn our attention to another one of the greatest discoveries, integral calculus. The integral is closely related to the derivative as we are about to see.

Antiderivatives and Indefinite Integrals

Up till now we have concerned ourselves with calculating the derivative $f(x)$ of a function $F(x)$:

$$f(x) = \frac{d}{dx} F(x) = F'(x)$$

But suppose instead of the function $F(x)$ what we have is its derivative $f(x)$. The process of finding $F(x)$ given its derivative $f(x)$ is called <u>integration</u> and we say that $F(x)$ is the <u>antiderivative</u> of $f(x)$. For example, given

$$f(x) = x^3$$

the antiderivative of $f(x)$ is:

$$F(x) = \frac{1}{4} x^4$$

because $F'(x) = f(x)$.

<u>Def</u> A function $F(x)$ is an <u>antiderivative</u> of a function $f(x)$ if $$F'(x) = f(x)$$ for all x in the domain of f.

Note an interesting thing about $F(x)$, it is not unique. If $F(x)$ is the antiderivative of $f(x)$, then

$$\frac{d}{dx} F(x) = f(x)$$

But note that if this true, then $F(x) + c$ is also an antiderivative of $f(x)$ where c is a constant:

$$\frac{d}{dx}\left[F(x)+c\right] = \frac{d}{dx} F(x) = f(x)$$

So $F(x)$ represents one of an infinite number of antiderivatives of $f(x)$, all of which differ by only a constant. We write the set of all antiderivatives of $f(x)$ as:

$$\boxed{\int f(x)\,dx = F(x) + c \qquad c = \text{arbitrary constant.}}$$

This is called the indefinite integral of $f(x)$. So, for example:

$$\int x^3\,dx = \frac{1}{4}x^4 + c$$

where the constant c is there because the indefinite integral represents all possible antiderivatives of $f(x)$.

> Def: The set of all antiderivatives of $f(x)$ is the indefinite integral of $f(x)$ with respect to x, denoted by
>
> $$\int f(x)\,dx\,,$$
>
> The symbol $\int$ is an integral sign. The function $f(x)$ is the integrand of the integral and x is the variable of integration.

The constant C is called the constant of integration. When we find $\int f(x)\,dx = F(x) + c$ we say we have integrated $f(x)$ or evaluated the integral.

Example

We know that

$$\frac{d}{dx}\cos x = -\sin x$$

So:

$$\int \sin x \, dx = -\cos x + C$$

Example

We know that

$$\frac{d}{dx}(x^3 + x) = 3x^2 + 1$$

So:

$$\int (3x^2+1)\, dx = x^3 + x + C$$

Example

It is easy to show that

$$\frac{d}{dx}\,\frac{3-2x}{3+2x} = -\frac{12}{(3+2x)^2}$$

So:

$$\int \left[-\frac{12}{(3+2x)^2}\right] dx = \frac{3-2x}{3+2x} + C$$

Some common integrals:

Derivative Formula	Indefinite Integral
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$
$\frac{d}{dx}x = 1$	$\int dx = \int 1\,dx = x + c$
$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right) = \sin kx$	$\int \sin kx\,dx = -\frac{\cos kx}{k} + c$
$\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$	$\int \cos kx\,dx = \frac{\sin kx}{x} + c$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}(-\cot x) = \csc^2 x$	$\int \csc^2 dx = -\cot x + c$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + c$
$\frac{d}{dx}(-\csc x)\,dx = \csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + c$

The column on the right is an example of a table of integrals. There is a more complete table of integrals in appendix I of the book (very last pages after the index). Even that is just a short table. There are tables of integrals in math handbooks that have hundreds of pages (for example, Table of Integrals, Series, and Products by I.S. Gradshteyn and I.M. Ryzhik has 1204 pages of which 843 pages are integral tables.

Rules for Indefinite Integrals

$$\int k f(x)\,dx = k \int f(x)\,dx \qquad k = \text{constant}$$

$$\int [-f(x)]\,dx = -\int f(x)\,dx$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Example

$$\int 4\sin x = 4\int \sin x\,dx$$

$$= -4\cos x + c$$

Example

(#6c, p.260)

$$\int \left(x^3 - \frac{1}{x^3}\right) dx = \int (x^3 - x^{-3})\,dx$$

$$= \frac{x^4}{4} - \frac{x^{-2}}{-2} + c$$

$$= \frac{x^4}{4} + \frac{1}{2x^2} + c$$

Example
(#12c, p. 280)

$$\int \left[\cos\frac{\pi x}{2} + \pi\cos x\right] dx = \int \cos\frac{\pi x}{2}\,dx + \int \pi\cos x\,dx$$

$$= \int \cos\frac{\pi x}{2}\,dx + \pi\int \cos x\,dx$$

$$= \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right) + \pi\sin x + c$$

Example
(#32, p. 280)

$$\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy = \frac{1}{7}\int dy - \int y^{-5/4}\,dy$$

$$= \frac{1}{7}y + 4\,y^{-1/4} + c$$

$$= \frac{y}{7} + \frac{4}{y^{1/4}} + c$$

Example

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \text{(trig identity)}$$

$$= \int \frac{1}{2}\,dx - \int \frac{1}{2}\cos 2x\,dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

Example

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c$$

Example

(#46, p. 281)

$$\int \frac{1}{2}(\csc^2 x - \csc x \cot x)\,dx = \frac{1}{2}\int (\csc^2 x - \csc x \cot x)\,dx$$

$$= \frac{1}{2}\left\{-\cot x + \csc x + c\right\}$$

$$= -\frac{1}{2}\cot x + \frac{1}{2}\csc x + c$$

Note: In the last line the constants were really $\frac{1}{2}c$. However, c is an arbitrary constant. Half an arbitrary constant is just another arbitrary constant, so we just absorb the $\frac{1}{2}$ into a new constant that I again called c.