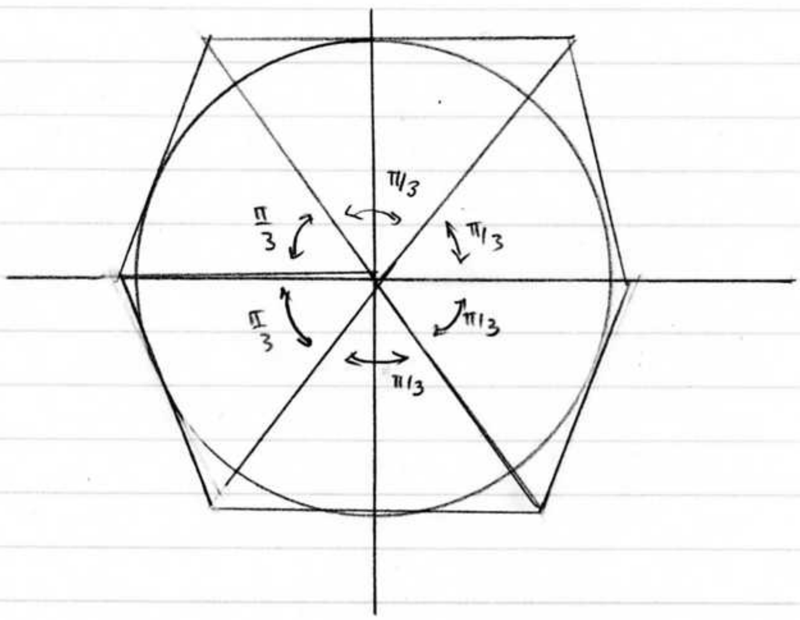
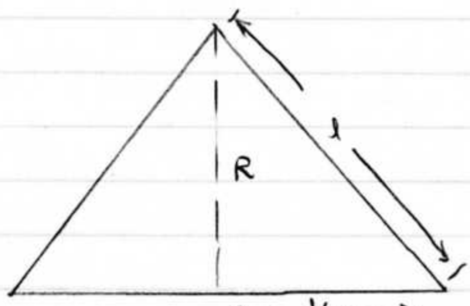
APPM 1350: Section 4.4: Estimating with Finite Sums

Consider a circle of radius R. We know from geometry that the area is πR^2. But where does that equation come from and more importantly, suppose we didn't know it? Then how would we calculate the area? One way is with finite sums. In particular, cut the circle into slices and inscribe those slices with triangles. So for example, using six slices:



My picture isn't perfect, but the idea is that we inscribe the circle inside these six identical equilaterial triangles with vertices of $60° = \frac{\pi}{3}$ rads.. The area of each triangle is half the base times the height. In our case the height is R.



and the sides have length l:

$$l^2 = \left(\frac{l}{2}\right)^2 + R^2$$

$$\Rightarrow \frac{3}{4}\ell^2 = R^2$$

$$\Rightarrow \ell = \frac{2R}{\sqrt{3}}$$

Thus the area of each triangle is:

$$A_\Delta = \frac{1}{2}R\ell$$

$$= \frac{1}{2}R\left(\frac{2R}{\sqrt{3}}\right)$$

$$= \frac{R^2}{\sqrt{3}}$$

Summing all six triangles the approximate area of the circle is:

$$\text{Area of circle} \approx 6A_\Delta = 2\sqrt{3}R^2 = 3.464R^2$$

Not $\pi R^2 \cong 3.1415R^2$, but not a bad approximation. And clearly as we use more triangles we would get a better and better approximation to the area of the circle. This is an example of <u>estimating with finite sums.</u>

Approximating the Area Under a Curve

If we are traveling at a constant velocity v_1 then the distance Δs_1 traveled over a time interval Δt_1 is

$$\Delta s_1 = v_1 \Delta t_1$$

Now suppose we change our velocity to v_2 for some interval Δt_2. The distance covered in that time increment will be

$$\Delta s_2 = v_2 \Delta t_2$$

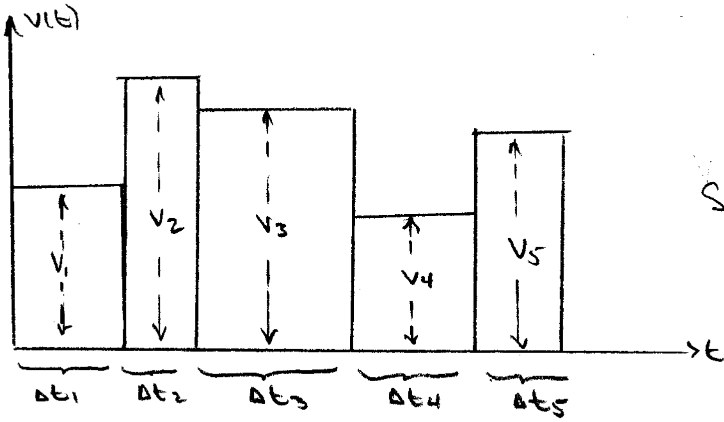
If we continue this over say n increments, then the total distance traveled over <u>all</u> the increments will be:

$$S_{TOTAL} = \Delta S_1 + \Delta S_2 + \cdots + \Delta S_n$$

$$= V_1 \Delta t_1 + V_2 \Delta t_2 + \cdots + V_n \Delta t_n$$

$$= \sum_{i=1}^{n} V_i \Delta t_i$$

The $\sum$ means "sum" and the $i=1$ and n mean the sum goes from $i=1$ to $i=n$ in increments of one. Graphically, if we plot $V(t)$ versus v, these sums represent the area of the rectangles with base Δt_i and height V_i; for example

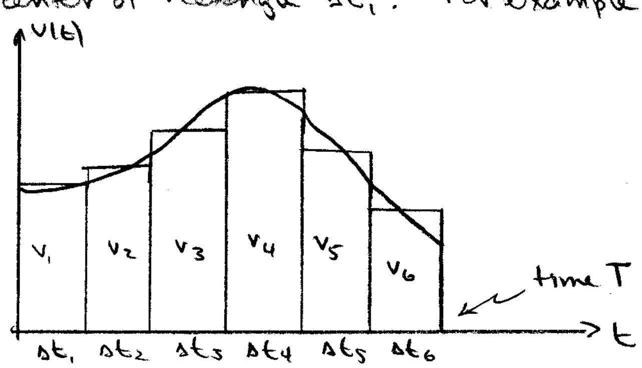


$$\Delta S_i = V_i \Delta t_i$$

$$S_{TOTAL} = \sum_{i=1}^{5} \Delta S_i$$

$$= \sum_{i=1}^{5} V_i \Delta t_i$$

Now suppose we are given some arbitrary curve of velocity as a function of time, Now cut time into increments Δt_i we can construct rectangles of height V_i where $V_i = V(t_i)$ and t_i is the time at the center of rectangle Δt_i. For example



Then the total distance traveled will approximately be:

$$S_{TOTAL} = \sum_{i=1}^{n} \Delta s_i$$

$$= \sum_{i=1}^{n} v_i(t_i)\,\Delta s_i$$

($n=6$ in the picture on the previous page)

where

$$t_1 = \frac{\Delta t_1}{2}$$

$$t_2 = \Delta t_1 + \frac{\Delta t_2}{2}$$

$$t_3 = \Delta t_1 + \Delta t_2 + \frac{\Delta t_3}{2}$$

$$\vdots$$

$$t_n = \sum_{i=1}^{n-1} \Delta t_i + \frac{\Delta t_n}{2}$$

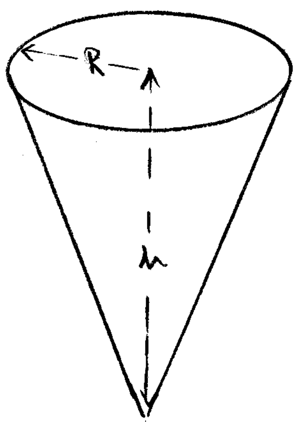
As we increase n (the number of rectangles) we will progressively improve our calculation of the total distance traveled over this time increment. Note that the the rectangles also approximate the area under the curve $v(t)$ between $t=0$ and some time $t=T$. So distance traveled between some time t_0 and time T is equal to the area under the curve $v(t)$ for $t \in [t_0, T]$.

This notion of approximating the area under a curve $f(x) \geq 0$ for $x \in [x_1, x_2]$ by summing rectangles is general and can be applied to any non-negative curve.

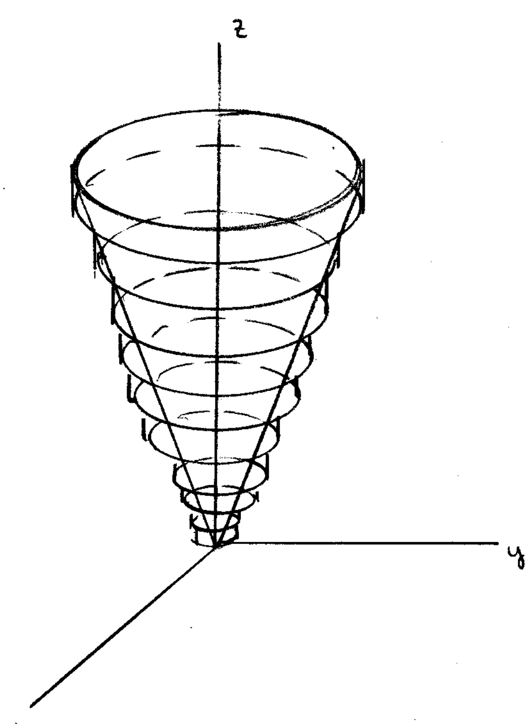
Approximating Volumes

We can extend the idea of approximating areas with finite sums to finding volumes. For example, consider a right circular cone of radius R and height h. The exact volume is

$$V = \frac{1}{3}\pi R^2 h \qquad (1)$$



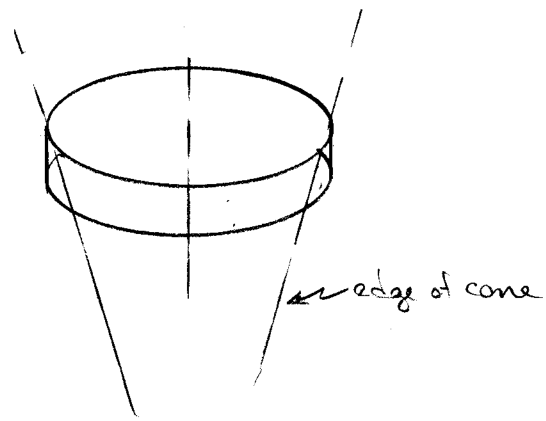
To estimate this volume we can sum up small right-circular cylinders. For example.



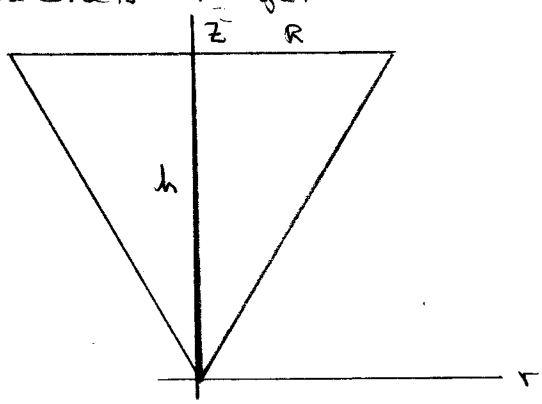
Each cylinder circumscribes a piece of the cone and has a volume

$$V_i = \pi r_i^2 \Delta z \qquad (2)$$

where we assume all the cylinders have the same height Δz. Now what is r_i?



r_i is the radius of the cone at the top of the ith cylinder. The cross-section of the cone is a triangle:



The radius R as a function of z is given by a straight line:

$$r(z) = \frac{R}{h} z$$

So

$$r_i = r(z_i) = \frac{R}{h} z_i \qquad (3)$$

where

$$\begin{aligned} z_1 &= \Delta z \\ z_2 &= 2\Delta z \\ &\vdots \\ z_i &= i\Delta z \end{aligned} \qquad (4)$$

and $i = 1, \ldots, n$ where n is given by

$$z_n = n \Delta z = h$$

$$\Rightarrow n = \frac{h}{\Delta z}$$

Note that Δz has to be choosen so that h is an integer multiple of h. More generally we would choose the number of cyclinders n and then <u>define</u>

$$\Delta z = \frac{h}{n} \qquad (5)$$

So collecting stuff, the volume of the ith cylinder is

$$V_i = \pi r_i^2 \Delta z \qquad \text{Using (2)}$$

$$= \pi \left(\frac{R}{h} z_i\right)^2 \Delta z \qquad \text{Using (3) for } r_i$$

$$= \pi \frac{R^2}{h^2} (i \Delta z)^2 \Delta z \qquad \text{Using (4) for } z_i$$

$$= \pi \frac{R^2 i^2}{h^2} (\Delta z)^3$$

$$= \frac{\pi R^2 i^2}{h^2} \left(\frac{h}{n}\right)^3 \qquad \text{Using (5) for } \Delta z$$

$$= \pi R^2 h \frac{i^2}{n^3}$$

and the total volume is approximately:

$$V_{approx} = \sum_{i=1}^{n} V_i$$

$$= \sum_{i=1}^{n} \pi R^2 h \frac{i^2}{n^3} \qquad (6)$$

We can factor the constants out of the sum, leaving:

$$V_{approx} = \frac{\pi R^2 h}{n^3} \sum_{i=1}^{n} i^2 \qquad (7)$$

It turns out that

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

so

$$V_{approx} = \frac{1}{3}\pi R^2 h \; \frac{(n+1)(2n+1)}{2n^2} \qquad (8)$$

Note that the true volume of a cone is $\frac{1}{3}\pi R^2 h$, so the error in our approximation involving cylinders is:

$$\begin{aligned} \epsilon &= V_{real} - V_{approx} \\ &= \frac{1}{3}\pi R^2 h \left[1 - \frac{(n+1)(2n+1)}{2n^2} \right] \\ &= \frac{1}{3}\pi R^2 h \left[\frac{\cancel{2n^2} - \cancel{2n^2} - n - 2n - 1}{2n^2} \right] \\ &= \frac{1}{3}\pi R^2 h \left[\frac{-3n-1}{2n^2} \right] \qquad (9) \end{aligned}$$

The relative error is:

$$\epsilon_{relative} = \frac{\epsilon}{V_{real}} = -\frac{1+3n}{2n^2}$$

Note that in the limit as we use "infinitely many" cylinders to approximate the volume of the cone (i.e., $n \to \infty$) that the relative error goes to zero!

Also note from (9) that the true error is negative. This means our approximate volume is always larger than the true volume (for finite n). This is obvious from the picture on page 5. However, had we choosen the radius r_i of the ith cylinder to be the radius of the cone at the height of the bottom of the cylinder then the approximate volume would now be less than the actual volume (the cylinders would be inscribed inside the cone).

The Average of a Nonnegative Function

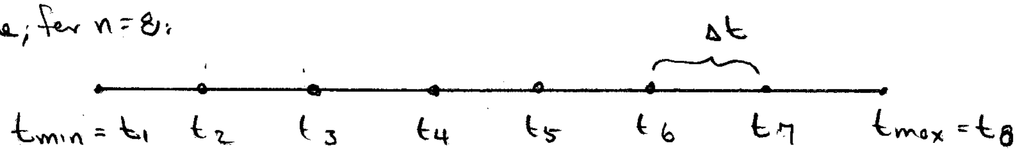
Let's return to the idea of some arbitrary curve $v(t)$ on the interval $[t_{min}, t_{max}]$. For example let $v(t)$ be the nonnegative velocity of an object over some interval of time. How would we calculate the average velocity during that interval? One way is to sample the velocity every Δt time increments and then compute the arithmetic average. So for example, assume we are going to same n times over $[t_{min}, t_{max}]$ where $n \geq 2$. Then

$$\Delta t = \frac{t_{max} - t_{min}}{n-1}$$

and we are sampling at times:

$$\begin{aligned} t_1 &= t_{min} \\ t_2 &= t_1 + \Delta t = t_{min} + \Delta t \\ t_3 &= t_2 + \Delta t = t_{min} + 2\Delta t \\ &\vdots \\ t_n &= t_{max} = t_{min} + (n-1)\Delta t \end{aligned}$$

For example, for $n=8$:



Then the average velocity would in general be:

$$V_{avg} \approx \frac{V(t_1) + V(t_2) + \cdots + V(t_n)}{n}$$

Note that this is really the <u>approximate average</u> velocity, approximate because its value depends on n.

As an example, consider $v(t) = t^3$ on $t = [0,1]$. Then using n samples:

$$\Delta t = \frac{1}{n-1}$$

and

$$t_i = 0 + (i-1)\Delta t$$

$$= \frac{i-1}{n-1}$$

and

$$V_i = v(t_i)$$

$$= \left(\frac{i-1}{n-1}\right)^3$$

So:

$$V_{avg} \approx \frac{1}{n}\sum_{i=1}^{n} v(t_i)$$

$$= \frac{1}{n(n-1)^3}\sum_{i=1}^{n}(i-1)^3$$

$$= \frac{1}{n(n-1)^3}\sum_{i=0}^{n-1} i^3 \qquad (10)$$

For $n = 3$, for example:

$$\Delta t = \frac{1-0}{2} = \frac{1}{2}$$

$$t_1 = 0$$
$$t_2 = \frac{1}{2}$$
$$t_3 = 1$$

$$V_1 = V(t_1) = 0$$
$$V_2 = V(t_2) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$
$$V_3 = V(t_3) = 1^3 = 1$$

and

$$V_{avg} \approx \frac{1}{3}\left(0 + \frac{1}{8} + 1\right)$$
$$= \frac{9}{3 \cdot 8}$$
$$= \frac{3}{8}$$

or using (1c):

$$V_{avg} \approx \frac{1}{3(2)^3}\left[0 + 1^3 + 2^3\right]$$
$$= \frac{1}{24}(1+8)$$
$$= \frac{3}{8}$$

This same approach to estimating the average of a function over an interval works for any function.

By the way,

$$\sum_{i=0}^{n-1} i^3 = \frac{(n-1)^2 n^2}{4}$$

so (10) we have been

$$V_{avg} \approx \frac{1}{n(n-1)^3} \frac{(n-1)^2 n^2}{4}$$

$$= \frac{n}{4(n-1)}$$

The "true" average velocity would be given by the limit as $n \to \infty$. Hence

$$\underset{(\text{true})}{V_{avg}} = \lim_{n \to \infty} \frac{n}{4(n-1)}$$

$$= \frac{1}{4} \lim_{n \to \infty} \frac{n}{n(1-\frac{1}{n})}$$

$$= \frac{1}{4} \lim_{n \to \infty} \frac{1}{(1-\frac{1}{n})}$$

$$= \frac{1}{4}$$

for $v(t) = t^3$ over $t \in [0,1]$.