APPM 1350: Section 4.5: Riemann Sums and Definite Integrals

Some Important Sums

Recall the sigma (Σ) notation from the last lecture. Formally,

Def The symbol $\sum_{k=1}^{n} a_k$ denotes the sum $a_1 + a_2 + \cdots + a_n$. The a's are the terms of the sum: a_1 is the first term, a_2 is the second term, a_k is the kth term, and a_n is the nth term. The variable k is the index of summation. The values of k run through the integers from 1 to n. The number 1 is the lower limit of summation; the number n is the upper limit of summation.

Example

$$\sum_{i=3}^{6} i^2 = 3^2 + 4^2 + 5^2 + 6^2 = 86$$

The following properties hold for finite sums:

1. Sum rule: $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. Difference rule: $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. Constant multiple rule: $\sum_{k=1}^{n} c\,a_k = c \sum_{k=1}^{n} a_k$ $\quad (c \in \mathbb{R})$

4. Constant value rule: $\sum_{k=1}^{n} c = nc$ $\quad (c \in \mathbb{R})$

These rules follow from the definition of the sigma notation and the normal rules for arithmetic,

Example

$$\sum_{n=1}^{n} (3k-k^2) = 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$$

Some particularly useful sums are for the first few powers of the integers.

$$\boxed{\begin{aligned} \sum_{k=1}^{n} k &= 1+2+\cdots+n = \frac{n(n+1)}{2} \\ \sum_{k=1}^{n} k^2 &= 1+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^{n} k^3 &= 1+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2 \end{aligned}}$$

An elegant proof of the first sum formula is due to Gauss. Write:

$$S = \sum_{k=1}^{n} k = 1+2+3+\cdots+n$$

Now write this down on two lines, but reverse the order of the summation on the second line and add the two lines "term-by-term":

$$\begin{aligned} S &= 1 + 2 + 3 + \cdots + n \\ S &= n + (n-1) + (n-2) + \cdots + 1 \\ \hline 2S &= n+1 + n+1 + n+1 + \cdots + n+1 \quad \text{n terms} \end{aligned}$$

$$\Rightarrow 2S = n(n+1) \qquad \text{since there are } n \text{ terms, each equal to } n+1$$

$$\Rightarrow S = \frac{n(n+1)}{2}$$

These formulas allow us to sum many series.

Example
(# 28, sec 4.5)

$$\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4} = \left(\sum_{k=1}^{7} k\right)^2 - \frac{1}{4}\sum_{k=1}^{7} k^3$$

$$= \left[\frac{7(8)}{2}\right]^2 - \frac{1}{4}\left[\frac{7(8)}{2}\right]^2$$

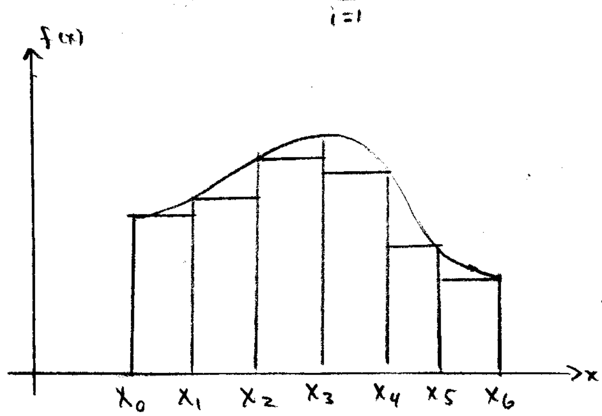
$$= (28)^2\left(1 - \frac{1}{4}\right)$$

$$= 588$$

Riemann Sums

In the previous lecture we discussed estimating the area under a curve $f(x)$ over some interval I by partitioning the interval into points $x_i \in I$ and computing the approximate area by something like

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x_i \qquad c_i \in [x_{i-1}, x_i]$$



Such a sum is called a Riemann sum of $f(x)$ for the interval I.

We also showed by example in the previous chapter that as we partition the interval finer and finer, the sum approaches a finite limit. In the limit as we partition the interval into an infinite number of intervals (i.e., $n \to \infty$) the limit is called the <u>definite integral</u> of f over I and write

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx$$

where

$$I = [a,b]$$

Note that as $n \to \infty$, ie; we partition the interval finer and finer the size of any Δx_i will get smaller. The <u>norm of the partition P of I</u> is defined as:

$$\|P\| = \min_{1 \le i \le n} \Delta x_i$$

So as $n \to \infty$ we also have $\|P\| \to 0$. Hence we could define the definite integral as:

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx$$

The two definitions imply each other and hence are equivalent. Formally;

<u>Def</u> Let $f(x)$ be a function defined on a closed interval $[a,b]$. We say that the <u>limit</u> of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ on $[a,b]$ as $\|P\| \to 0$ is the number S if the following condition is satisfied:

Given any $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that for <u>every</u> partition P of $[a,b]$

$$\|P\| < \delta \quad \Rightarrow \quad \left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - S \right| < \epsilon$$

for any choice of the numbers c_k in the subintervals $[x_{k-1}, x_k]$.

When a definite integral of $f(x)$ over I exists, we say that $f(x)$ is <u>integrable</u> over I and we say the Riemann sums <u>converge</u> to the definite integral. So what makes a function integrable?

> <u>Theorem</u> All continuous functions are integrable. That is, if a function f is continuous on an interval $[a,b]$, then its definite integral over $[a,b]$ exists.

<u>More generally any piecewise continuous function is integrable</u>. Some discontinuous functions are also integrable. The key is whether the limit of the Riemann sums exists, or more colloquially whether we can define the "area under the curve".

<u>Terminology</u>

upper limit of integration

integrand

integral sign $\rightarrow$ $$\int_a^b f(x)\,dx$$ $\leftarrow$ variable of integration

lower limit of integration

and the whole thing is called the <u>definite integral of $f(x)$ from a to b.</u> Note that in definite integration (but <u>not</u> in indefinite integration), the <u>variable of integration is arbitrary</u>. It is called a <u>dummy variable.</u> So

$$\int_a^b f(x)\,dx = \int_a^b f(\alpha)\,d\alpha = \int_a^b f(\text{susan})\,d(\text{susan})$$

In the last case using "susan" as a variable may seem odd (okay, it is odd), but it is legal. These are just symbols that tell us what we are integrating with respect to.

Area Under the Graph of a Nonnegative Function

Up till now we have consider Riemann sums for nonnegative functions, i.e., $f(x) \geq 0$. Then clearly:

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k = \int_a^b f(x)\,dx \geq 0$$

But what about nonnegative functions? In particular, suppose we have some $g(x) \leq 0$ on I? Then:

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} g(c_k) \Delta x_k = \int_a^b g(x)\,dx \leq 0$$

since $g(c_k) \leq 0$ and $\Delta x_k > 0$. In terms of "area under the curve", what we just showed is that the <u>area is positive if $f(x) \geq 0$ on I and negative if $f(x) \leq 0$ on I.</u> This is in effect a definition of what we mean by "area under a curve". So:

$$\boxed{\begin{array}{ll} \displaystyle\int_a^b f(x)\,dx \geq 0 & \text{if } f(x) \geq 0 \text{ on } [a,b] \\[2ex] \displaystyle\int_a^b f(x)\,dx \leq 0 & \text{if } f(x) \leq 0 \text{ on } [a,b] \end{array}}$$

Example

(#30, sec 4.5)

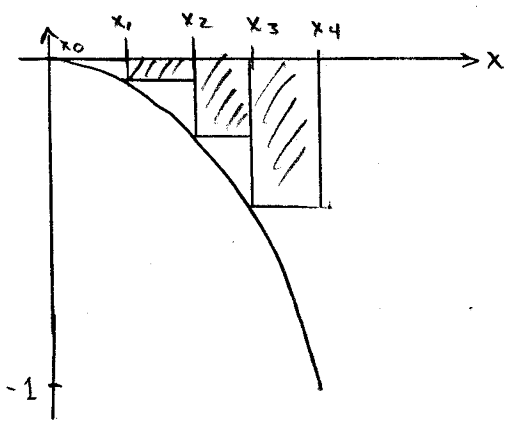
Partition $f(x) = -x^2$ on $[0,1]$ into four subintervals of equal length. Graph the function with the rectangles associated with the Riemann sum

$$\sum_{k=1}^{4} f(c_k)\,\Delta x_k .$$

given that c_k is a) the left-hand endpoint, b) the right-hand endpoint, c) the midpoint of the kth subinterval.

Solution:

a)



$x_0 = 0$

$x_1 = 1/4$ $\qquad \Delta x_i = \Delta x = \frac{1}{4}$

$x_2 = 1/2$

$x_3 = 3/4$

$x_4 = 1$

$c_1 = 0$

$c_2 = 1/4$

$c_3 = 1/2$

$c_4 = 3/4$

$$\sum_{k=1}^{4} f(c_k)\Delta x_k = -(0)^2\left(\tfrac{1}{4}\right) - \left(\tfrac{1}{4}\right)^2\left(\tfrac{1}{4}\right) - \left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{4}\right) - \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right)$$

$$= -\frac{1}{64} - \frac{1}{16} - \frac{9}{64}$$

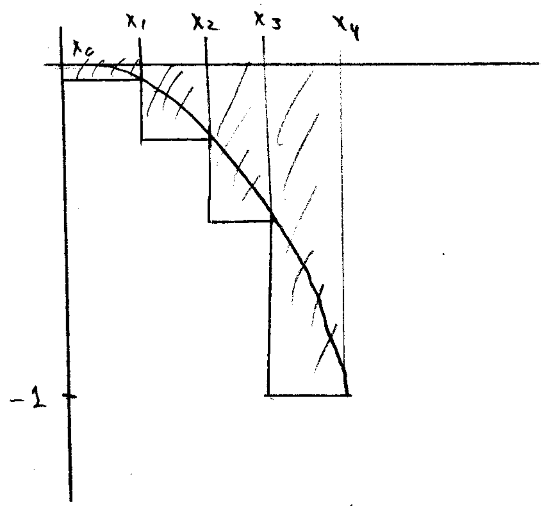
$$= -\frac{1+4+9}{64}$$

$$= -\frac{14}{64}$$

$$= -\frac{7}{32}$$

$$\approx -0.219$$

b)



$x_0 = 0$

$x_1 = \frac{1}{4}$

$x_2 = \frac{1}{2}$

$x_3 = \frac{3}{4}$

$x_4 = 1$

$\Delta x_i = \Delta x = \frac{1}{4}$

$c_1 = \frac{1}{4}$

$c_2 = \frac{1}{2}$

$c_3 = \frac{3}{4}$

$c_4 = 1$

$$\sum_{k=1}^{4} f(c_k)\Delta x_k = -\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right) - \left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right) - \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) - 1^2\left(\frac{1}{4}\right)$$

$$= -\frac{1}{4}\left[\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right]$$

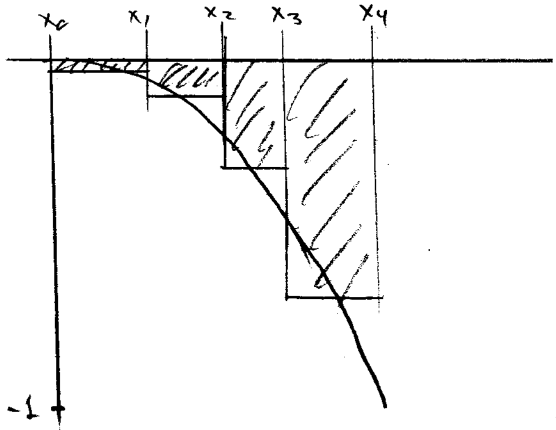
$$= -\frac{1}{4}\left[\frac{1+4+9+16}{16}\right]$$

$$= -\frac{30}{64}$$

$$= -\frac{15}{32}$$

$$\approx -0.469$$

c)



$x_0 = 0$

$x_1 = \frac{1}{4}$

$x_2 = \frac{1}{2}$ $\qquad \Delta x_i = \Delta x = \frac{1}{4}$

$x_3 = \frac{3}{4}$

$x_4 = 1$

$c_1 = \frac{1}{8}$

$c_2 = \frac{3}{8}$

$c_3 = \frac{5}{8}$

$c_4 = \frac{7}{8}$

$$\sum_{k=1}^{4} f(c_k)\,\Delta x_k = -\left(\frac{1}{8}\right)^2\left(\frac{1}{4}\right) - \left(\frac{3}{8}\right)^2\left(\frac{1}{4}\right) - \left(\frac{5}{8}\right)^2\left(\frac{1}{4}\right) - \left(\frac{7}{8}\right)^2\left(\frac{1}{4}\right)$$

$$= -\frac{1}{4}\left(\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64}\right)$$

$$= -\frac{1}{4}\left(\frac{84}{64}\right)$$

$$= -\frac{21}{64}$$

$$\simeq -0.328$$

Constant Functions

If $f(x)$ has a constant value c on $[a,b]$, then

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a)$$

This result is obvious if you think about the definite integral as a Riemann sum. In this case the value is the same for all partitions, including just a single subinterval.

Example

(#44, sec 4.5)

$$\int_3^7 (-20)\,dx = -20\ (7-3)$$

$$= -20(4)$$

$$= -80$$

Evaluating Definite Integrals

The previous lecture actually evaluated some Riemann limits as $\|P\| \to 0$, hence definite integrals. Here we finish up with another example.

Example

(#64, sec 4.5)

$$\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr = \lim_{\|P\|\to 0} \sum_{k=1}^{n} R_k \Delta r_k$$

where P is some partition of $[\sqrt{2}, 5\sqrt{2}]$ into r_i
$\sqrt{2} = r_0 < r_1 < r_2 < \cdots < r_n = 5\sqrt{2}$
and $r_{k-1} \leq R_k \leq r_k$ is some value of r (the integrand) in the subinterval $[r_{k-1}, r_k]$.

Let us take n subintervals of equal length:

$$\Delta r = \frac{5\sqrt{2} - \sqrt{2}}{n}$$

$$= \frac{4\sqrt{2}}{n} \qquad (1)$$

We can choose R_k to be <u>any</u> point in $[r_{k-1}, r_k]$. So arbitrarily let's use the left-hand endpoint. Then:

$$\begin{aligned}
r_0 &= \sqrt{2} \\
r_1 &= \sqrt{2} + \Delta r \\
r_2 &= \sqrt{2} + 2\Delta r \\
&\vdots \\
r_k &= \sqrt{2} + (k-1)\Delta r \\
&\vdots \\
r_n &= \sqrt{2} + n\Delta r
\end{aligned} \qquad (2)$$

So

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} R_k \Delta r_k \\
&= \sum_{k=1}^{n} \left(\sqrt{2} + (k-1)\Delta r\right)\Delta r \\
&= \sum_{k=1}^{n} \sqrt{2}\,\Delta r + \sum_{k=1}^{n} (k-1)\Delta r^2 \\
&= n\sqrt{2}\,\Delta r + \Delta r^2 \sum_{k=1}^{n} k - n\Delta r^2 \\
&= n\sqrt{2}\,\Delta r + \Delta r^2 \frac{n(n+1)}{2} - n\Delta r^2 \\
&= 8 + 16\frac{n+1}{n} - \frac{32}{n}
\end{aligned}$$

Then:

$$\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr = \operatorname*{limit}_{\|P\| \to 0} \sum_{k=1}^{n} R_k \Delta r_k = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(8 + 16\frac{n+1}{n} - \frac{32}{n}\right) = 8 + 16 = 24$$