APPM 1350: Section 6.9: Derivatives of Inverse Trigonometric Functions; Integrals

Derivatives of Inverse Trig Functions

The derivatives of inverse trig functions can be derived from implicit differentiation of the associated trig function. For example, let

$$y = \sin^{-1} x$$

Note that the range of $\sin^{-1} x$ is $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$. Note also that the domain is $-1 \le x \le 1$

$$\Rightarrow \sin y = x$$

$$\Rightarrow \cos y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

Note that $\cos y > 0$ for $-\pi/2 \le y \le \pi/2$
$\Rightarrow \cos y = \sqrt{1 - \sin^2 y}$

$$= \frac{1}{\sqrt{1 - \sin^2 y}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

where $|x| < 1$ to avoid division by zero

Hence

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} \qquad |x| < 1$$

and if $u = u(x)$, then

$$\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx} \qquad |u| < 1$$

by the chain rule.

The derivation of the derivatives of the other inverse trig functions follows in a similar manner. The results are:

1. $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ $\quad |u| < 1$

2. $\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ $\quad |u| < 1$

3. $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

4. $\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx}$

5. $\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$ $\quad |u| > 1$

6. $\frac{d}{dx}\csc^{-1}u = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$ $\quad |u| > 1$

Example

(# 1, see 6.9)

$$\frac{d}{dx}\cos^{-1}(x^2) = -\frac{1}{\sqrt{1-x^4}}\frac{d}{dx}x^2$$

$$= -\frac{2x}{\sqrt{1-x^4}}$$

$\left\{ \text{since } \frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \right.$

Example

(#14, sec 6.9)

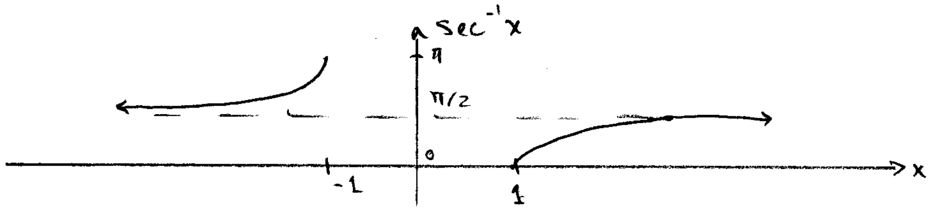
$$\frac{d}{dx}\tan^{-1}(\ln x) = \frac{1}{1+(\ln x)^2}\frac{d}{dx}\ln x \qquad \left\{ \text{since } \frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx} \right.$$

$$= \frac{1}{x[1+(\ln x)^2]}$$

Example

(#66, sec 6.9) Find $\displaystyle \lim_{x\to 1^+} \frac{\sqrt{x^2-1}}{\sec^{-1}x}$

Solution:



Since $\displaystyle \lim_{x\to 1^+} \sec^{-1}x = 0$, then $\displaystyle \lim_{x\to 1^+} \frac{\sqrt{x^2-1}}{\sec^{-1}x}$ is indeterminate $\left(\frac{0}{0}\right)$.

Thus use L'Hôspital's rule:

$$\lim_{x\to 1^+} \frac{\sqrt{x^2-1}}{\sec^{-1}x} = \lim_{x\to 1^+} \frac{\left(\frac{x}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{|x|\sqrt{x^2-1}}\right)}$$

$$= \lim_{x\to 1^+} x\,|x|$$

$$= 1$$

Integrations That Evaluate to Inverse Trig Functions

The table on page 2 can be read backward to give the antiderivatives of $\frac{1}{\sqrt{1-x^2}}$, $\frac{1}{1+x^2}$, and $\frac{1}{x\sqrt{x^2-1}}$. Specifically:

$$\boxed{\begin{array}{ll} \displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c & |x|<1 \\[2ex] \displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c & x\in\mathbb{R} \\[2ex] \displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + c & |x|>1 \end{array}}$$

We can generalize these somewhat. For example; letting $x^2 < a^2$ and $a \neq 0$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{1}{\sqrt{a^2}} \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$$= \frac{1}{\sqrt{a^2}} \int \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

Let $u = \frac{x}{a} \Rightarrow du = \frac{dx}{a}$

$$= \frac{a}{|a|} \int \frac{du}{\sqrt{1-u^2}}$$

Assume $a>0$. This is okay because a shows up via a^2 in the integral. So we can take a to be the positive square root of a^2

$$= \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} u + c$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + c$$

The integrals on the previous page thus generalize to:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + c \qquad \text{if } u^2<a^2 \text{ and } a\neq 0$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c \qquad a\neq 0$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c \qquad \text{if } u^2>a^2 \text{ and } a\neq 0$$

<u>Example</u>
(#31, sec 6.9)

$$\int_0^2 \frac{dt}{8+2t^2} = \frac{1}{2}\int_0^2 \frac{dt}{4+t^2} \qquad \Rightarrow a=2$$

$$= \frac{1}{2}\cdot\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]\Bigg|_0^2$$

$$= \frac{1}{4}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{16}$$

Note this could also have been done by substitution:

$$\int_0^2 \frac{dt}{8+2t^2} = \frac{1}{\sqrt{2}}\int_0^{2\sqrt{2}} \frac{du}{8+u^2}$$

Let $u=\sqrt{2}\,t$
$\Rightarrow du = \sqrt{2}\,dt$

$$= \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{8}}\tan^{-1}\left(\frac{u}{\sqrt{8}}\right)\Bigg|_0^{2\sqrt{2}}$$

$$= \frac{1}{\sqrt{16}}\tan^{-1} 1 = \frac{\pi}{16}$$

Example

(#41, sec 6.9)

$$\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{1+(\sin\theta)^2} = 2\int_{-1}^{1} \frac{du}{1+u^2}$$

Let $u = \sin\theta$
$du = \cos\theta\, d\theta$

$$= 2\tan^{-1} u \Big|_{-1}^{1}$$

$$= 2\left(\tan^{-1} 1 - \tan^{-1}(-1)\right)$$

$$= 4\tan^{-1} 1$$

since $\tan^{-1} x$ is an odd function

$$= 4\,\frac{\pi}{4}$$

$$= \pi$$

Example

(#54, sec 6.9)

$$\int_{2}^{4} \frac{2\, dx}{x^2-6x+10} = \int_{2}^{4} \frac{2\, dx}{1+(x^2-6x+9)}$$

$$= 2\int_{2}^{4} \frac{dx}{1+(x-3)^2}$$

clever, huh.

$$= 2\int_{-1}^{1} \frac{dx}{1+u^2}$$

Let $u = x-3$
$du = dx$

$$= 2\tan^{-1} u \Big|_{-1}^{1}$$

$$= 2\left(\tan^{-1} 1 - \tan^{-1}(-1)\right)$$

$$= 4\,\frac{\pi}{4}$$

$$= \pi$$

Example

(#58, sec 6.9)

$$\int \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx = -\int e^u\,du$$

Let $u = \cos^{-1}x$

$\Rightarrow du = -\dfrac{dx}{\sqrt{1-x^2}}$

$$= -e^u + c$$

$$= -e^{\cos^{-1}x} + c$$

Example

(#74, sec 6.9) Solve the IVP $\dfrac{dy}{dx} = \dfrac{1}{x^2+1} - 1$; $y(0) = 1$

Solution

$$\frac{dy}{dx} = \frac{1}{x^2+1} - 1$$

$$\Rightarrow dy = \left(\frac{1}{x^2+1} - 1\right) dx$$

$$\Rightarrow \int dy = \int \frac{dx}{x^2+1} - \int dx$$

$$\Rightarrow y(x) = \tan^{-1}x - x + c$$

Then since $y(0) = 1$ we have

$$1 = \cancel{\tan^{-1}0}^{\,0} - 0 + c$$

$$\Rightarrow c = 1$$

Hence: $y(x) = \tan^{-1}x - x + 1$ is the solution to the IVP.

Note: One can also solve these kind of IVPs by putting the initial value in as the lower limit of integration. So given

$$\frac{dy}{dx} = f(x) \quad ; \quad y(x_0) = y_0$$

$$\Rightarrow \quad dy = f(x)\,dx$$

$$\Rightarrow \int_{y_0}^{y} dy = \int_{x_0}^{x} f(x)\,dx$$

The upper limits of integration are x and y

$$\Rightarrow y\Big|_{y_0}^{y} = F(x) - F(x_0)$$

where $F(x) = \int f(x)\,dx$

$$\Rightarrow y(x) = y_0 + F(x) - F(x_0)$$

So in the previous example;

$$\frac{dy}{dx} = \frac{1}{x^2+1} - 1 \qquad y(0) = 1$$

$$\Rightarrow \int_1^y dy = \int_0^x \left(\frac{1}{x^2+1} - 1\right) dx$$

$$\Rightarrow y(x) - 1 = \left(\tan^{-1}x - x\right)\Big|_0^x$$

$$= \tan^{-1}x - x$$

$$\Rightarrow y(x) = \tan^{-1}x - x + 1$$

as before. Use whatever technique you prefer. Most math books seem to do these IVPs as on the previous page with the explicit constant. Most physics books seem to do them the way I showed on this page.