APPM 1350: Exam 1: Solutions

1. Is the function even, odd, or neither?

a) $y = (-x)^{2/3}$ $\boxed{\text{Even}}$

Why? $y = (-x)^{2/3} = \left[(-x)^2\right]^{1/3} \Rightarrow y(-x) = \left[(-(-x))^2\right]^{1/3} = \left[x^2\right]^{1/3} = y(x)$

b) $y = \dfrac{\sin x}{x^2-1}$ $\boxed{\text{Odd}}$

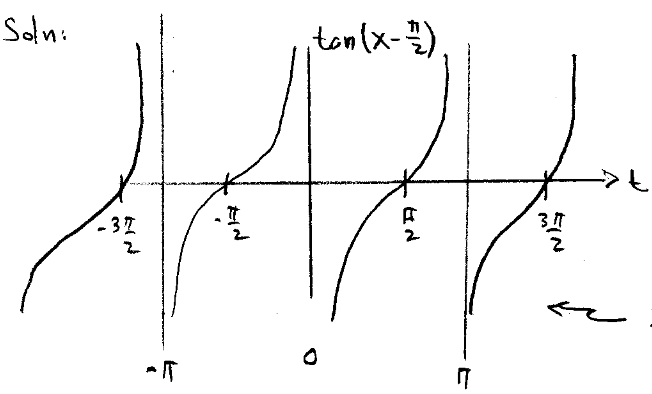
Why? $y(-x) = \dfrac{\sin(-x)}{(-x)^2-1} = \dfrac{-\sin x}{x^2-1} = -\dfrac{\sin x}{x^2-1} = -y(x)$

c) $y = 1$ $\boxed{\text{Even}}$

Why? $y(-x) = 1 = y(x)$

2. a) Find the domain of $f(x) = \tan(x - \frac{\pi}{2})$

Soln:

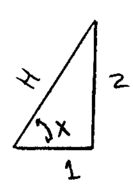


$$\boxed{D = \{x \in \mathbb{R} : x \neq n\pi,\ n \in \mathbb{Z}\}}$$

$\tan(x - \frac{\pi}{2})$ is just $\tan x$ shifted $\frac{\pi}{2}$ to the right

b) If $x \in [0, \frac{\pi}{2}]$ and $\tan x = 2$, find $\sin x$

Soln:



$\tan x = 2$ means opposite over adjacent is equal to 2. So we have the triangle to the left. The hypotenuse is then $H = \sqrt{1^2+2^2} = \sqrt{5}$ and so

$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{2}{\sqrt{5}}$$

$$\boxed{\sin x = \frac{2}{\sqrt{5}}}$$

3. Calculate the following limits. If they do not exist, write DNA

a) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$ Soln: $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}$

$$= \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)}$$

$$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3}$$

$$= \frac{1}{3+3}$$

$$= \boxed{\frac{1}{6}}$$

b) $\lim_{x\to 2} \sqrt{5+\sqrt[3]{2x^5}} = \sqrt{5+\sqrt[3]{2(2)^5}} = \sqrt{5+(2)^{6/3}} = \sqrt{5+4} = \boxed{3}$

c) $\lim_{x\to 1} \cos\left(\frac{x^2-2x+1}{x^2-1}\right) = \lim_{x\to 1} \cos\left(\frac{(x-1)^2}{(x-1)(x+1)}\right)$

$$= \lim_{x\to 1} \cos\left(\frac{x-1}{x+1}\right)$$

$$= \cos 0$$

$$= \boxed{1}$$

d) $\lim_{x\to 0} \frac{x+5}{x} = \lim_{x\to 0} \left(1+\frac{5}{x}\right)$ $\boxed{\text{does not exist}}$

Why? $\lim_{x\to 0^-} \left(1+\frac{5}{x}\right) = -\infty$

$\lim_{x\to 0^+} \left(1+\frac{5}{x}\right) = +\infty$

Left and right hand limits differ $\Rightarrow$ no limit

4. a) State the Intermediate Value Theorem (for continuous functions).

Soln:

$\boxed{\text{If } f(x) \text{ is continuous on } [a,b] \text{ then if } y_0 \text{ is a number between } f(a) \text{ and } f(b) \text{, there will be a point } x_0 \in (a,b) \text{ such that } y_0 = f(x_0)}$

b) Explain why the equation $\cos x = x$ has a solution in $[0, \frac{\pi}{2}]$

Soln:

If $\cos x = x$, then $f(x) = \cos x - x = 0$. So we want to show $f(x) = \cos x - x$ has a root for some $x_0 \in [0, \frac{\pi}{2}]$. Note that $f(x)$ is continuous on $[0, \frac{\pi}{2}]$. Further, $f(0) = 1$ and $f(\frac{\pi}{2}) = -\pi/2$. Since $y_0 = f(0) = 0$ lies between $f(0)$ and $f(\frac{\pi}{2})$, there is some point $x_0 \in (0, \frac{\pi}{2})$ where $f(x_0) = 0$ by the intermediate value theorem.

c) Does the function $f(x) = \sqrt{x}\cos\left(\frac{9}{x}\right)$ have a continuous extension at $x = 0$?

Soln:

The domain of $f(x)$ is $\{x : x \in \mathbb{R} \text{ and } x > 0\}$. To make $f(x)$ right continuous at $x = 0$ we must define a point $f(0) = \lim_{x \to 0^+} f(x)$. So does the limit exist? Note that

$$0 \le \left|\sqrt{x}\cos\left(\frac{9}{x}\right)\right| = \sqrt{x}\,\left|\cos\left(\frac{9}{x}\right)\right| \le \sqrt{x}$$

Since $\lim_{x \to 0^+} 0 = 0$ and $\lim_{x \to 0^+} \sqrt{x} = 0$, by the sandwich theorem

$$\lim_{x \to 0^+} \sqrt{x}\cos\left(\frac{9}{x}\right) = 0$$

So the function

$$f_{new}(x) = \begin{cases} \sqrt{x}\cos\left(\frac{9}{x}\right) & x > 0 \\ 0 & x = 0 \end{cases}$$

is continuous. So, yes, $f(x)$ has a continuous extension.

5. On what intervals is each of the following functions differentiable? In each case, calculate the derivative where it exists.

a) $y = (x^2+1)(x+5+\frac{1}{x})$ Differentiable all $x \in \mathbb{R}$ except $x=0$

$$\Rightarrow y' = (x^2+1)\frac{d}{dx}(x+5+x^{-1}) + (x+5+x^{-1})\frac{d}{dx}(x^2+1)$$

$$= (x^2+1)(1-x^{-2}) + (x+5+x^{-1})(2x)$$

$$= x^2 - 1 + 1 - x^{-2} + 2x^2 + 10x + 2$$

$$= \boxed{3x^2 + 10x - \frac{1}{x^2} + 2 \quad (x \neq 0)}$$

(Differentiable all $x \in \mathbb{R}$ except $x=0$)

b) $y = |x-4|$

$$\Rightarrow y = \begin{cases} 4-x & x<4 \\ x-4 & x \geq 4 \end{cases}$$

Now, $\frac{d}{dx}(4-x) = -x$ and $\frac{d}{dx}(x-4) = x$.

But!!

Note that $\lim_{x\to 4} y'(x)$ does not exist $\Rightarrow$ $y(x)$ is differentiable at all $x \in \mathbb{R}$ except $x=4$.
and
$$y'(x) = \begin{cases} -x & x<4 \\ x & >4 \end{cases}$$

c) $y = 5\cos\left(\frac{\pi}{3}\right)$

$y'(x) = 0$ since $y(x)$ is a constant. $\Rightarrow$ $y(x)$ is differentiable for all $x \in \mathbb{R}$ and $y'(x) = 0$.

6. a) Using the definition, calculate the derivative of $f(x) = \frac{1}{x+3}$

Soln: $\frac{d}{dx}\left(\frac{1}{x+3}\right) = \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{x+h+3} - \frac{1}{x+3}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x+3}{(x+h+3)(x+3)} - \frac{x+3+h}{(x+h+3)(x+3)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-h}{(x+h+3)(x+3)}\right]$$

$$= \lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)}$$

$$= \boxed{-\frac{1}{(x+3)^2} \qquad (x \neq -3)}$$

b) Evaluate $f'(0)$

Soln: $f'(0) = -\frac{1}{(0+3)^2} = \boxed{-\frac{1}{9}}$

c) Find the equation of the tangent line to the graph of f at the point $(0, \frac{1}{3})$.

Soln: $f(0) = \frac{1}{3}$

$f'(0) = -\frac{1}{9}$

$\Rightarrow y_t(x) = \frac{1}{3} - \frac{1}{9}(x-0)$

$\Rightarrow \boxed{y_t(x) = \frac{1}{3} - \frac{x}{9}}$

7. A rock climber accidently kicks a rock loose when she is 576 feet high.

a) How long does it take the rock to hit the ground?

Soln: $S = \frac{1}{2} g t^2$ $\qquad g = 32.2 \text{ ft/s}^2$

$= \frac{1}{2}(32.2) t^2$

Thus

$576 = \frac{1}{2}(32.2) t^2$

$$\Rightarrow \boxed{t_I = \sqrt{\frac{2(576)}{32.2}} \text{ seconds}} \quad \sim 6 \text{ sec}$$

b) What is the average velocity at this time?

Soln: $V_{avg} = \frac{\Delta s}{\Delta t} = \frac{s(t_I) - s(0)}{t_I - 0}$

$$= \frac{576 - 0}{\sqrt{\frac{2(576)}{32.2}} - 0}$$

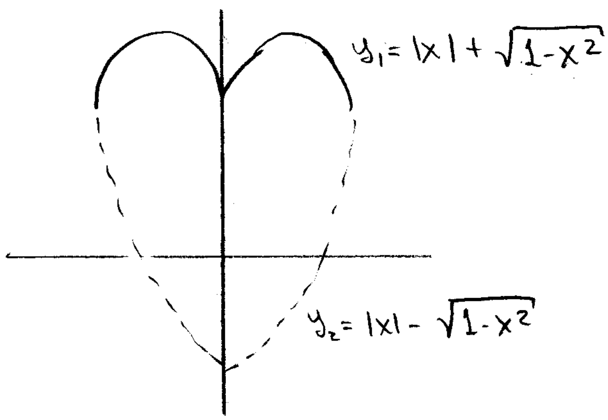
$$= \boxed{\sqrt{\frac{(576)(32.2)}{2}} \ \frac{\text{ft}}{\text{sec}}} \quad \sim 96 \frac{\text{ft}}{\text{s}}$$

c) What is the instantaneous velocity of the rock at the moment it hits the ground?

Soln: $V(t) = \frac{ds}{dt} = gt = (32.2)t$

$$\Rightarrow \boxed{V(t_I) = 32.2 \sqrt{\frac{2(576)}{32.2}} \ \frac{\text{ft}}{\text{sec}}} \quad \sim 192 \frac{\text{ft}}{\text{s}}$$

<u>Joker problem</u> Imagine your heart as the contour given by the union of the graphs of two functions $y = |x| + \sqrt{1-x^2}$ (solid line) and $y = |x| - \sqrt{1-x^2}$ (dashed line) for $-1 \le x \le 1$. Cupid always shoots arrows horizontally towards your heart. If he hits it centrally, you fall in love. If he hits it tangentially, you get blue. If he misses, he won't next time. Can you tell at what point(s) on the contour has your heart been hit if you start to feel blue?



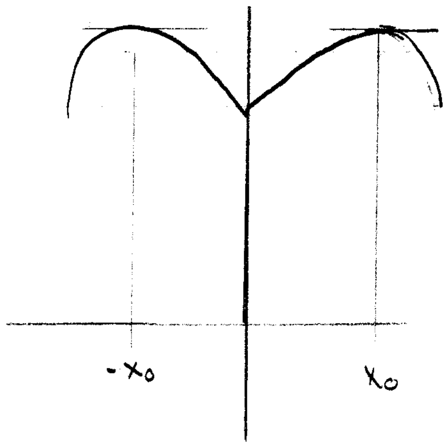
Solution:

If you feel blue it means Cupid's arrow struck tangentially. Since he is shooting horizontally, hitting tangentially means the slope of the tangent line (along his arrow) is zero at the point where the arrow strikes (or grazes) your heart. So, the slope of the tangent line to $y = |x| + \sqrt{1-x^2}$ is given by the derivative:

$$\frac{d}{dx}\, |x| + \sqrt{1-x^2} = \frac{d}{dx} \begin{cases} -x + \sqrt{1-x^2} & x < 0 \\ \\ x + \sqrt{1-x^2} & x > 0 \end{cases}$$

Note that the derivative doesn't exist at $x=0$ because $|x|$ is not differentable at $x=0$ (hence y_1 has a cusp at $x=0$).

Graphically, the tangent line is zero where the slope is zero. This



occurs at $\pm x_0$ as shown here ($x_0 > 0$ and $0 < x_0 < 1$)

For the lower contour the only possibility for a tangental hit would be at $x=0$. But, $y_2 = |x| - \sqrt{1-x^2}$ is not differentiable at $x=0$, hence there is no tangent line at $x=0$. Thus the bottom of your heart cannot be struck tangentially.

So... Your heart was struck at x_0 on the top curve as graphed above. We need to compute the derivative to determine x_0. It is sufficient to consider the $0 \le x < 1$ case

$$\frac{d}{dx}|x| + \sqrt{1-x^2} = \frac{d}{dx} x + \sqrt{1-x^2} \qquad 0 < x < 1$$

$$= 1 + \frac{d}{dx}\sqrt{1-x^2}$$

$$= 1 + \lim_{h \to 0} \frac{\sqrt{1-(x+h)^2} - \sqrt{1-x^2}}{h}$$

$$= 1 + \lim_{h \to 0} \frac{\left(\sqrt{1-(x+h)^2} - \sqrt{1-x^2}\right)\left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2}\right)}{h\left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2}\right)}$$

$$= 1 + \lim_{h \to 0} \frac{1-(x+h)^2 - (1-x^2)}{h\left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2}\right)}$$

$$= 1 + \lim_{h \to 0} \frac{\cancel{1} - \cancel{x^2} - 2xh - h^2 - \cancel{1} + \cancel{x^2}}{h\left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2}\right)}$$

$$= 1 + \lim_{h \to 0} \frac{h(-2x-h)}{h\left(\sqrt{1-(x+h)^2} + \sqrt{1-x^2}\right)}$$

$$= 1 + \lim_{h \to 0} \frac{-2x-h}{\sqrt{1-(x+h)^2} + \sqrt{1-x^2}}$$

$$= 1 - \frac{2x}{2\sqrt{1-x^2}}$$

$$= 1 - \frac{x}{\sqrt{1-x^2}}$$

Setting this to zero we have

$$0 = 1 - \frac{x_0}{\sqrt{1-x_0^2}}$$

$$\Rightarrow 1 = \frac{x_0}{\sqrt{1-x_0^2}}$$

$$\Rightarrow \sqrt{1-x_0^2} = x_0$$

$$\Rightarrow 1 - x_0^2 = x_0^2$$

$$\Rightarrow 2x_0^2 = 1$$

$$\Rightarrow x_0 = \frac{1}{\sqrt{2}}$$

(we only use the positive square root because by assumption $0 < x_0 < 1$)

Thus, the arrow struck at $x = \pm x_0$, hence

$$x = \pm x_0$$
$$= \pm \frac{1}{\sqrt{2}}$$

The value of the function at these points is:

$$y_1\left(\pm \frac{1}{\sqrt{2}}\right) = \left|\pm \frac{1}{\sqrt{2}}\right| + \sqrt{1-\left(\pm\frac{1}{\sqrt{2}}\right)^2}$$
$$= \frac{1}{\sqrt{2}} + \sqrt{1-\frac{1}{2}}$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}}$$
$$= \sqrt{2}$$

Hence the points of interest are

$$\boxed{\left(-\frac{1}{\sqrt{2}}, \sqrt{2}\right) \text{ and } \left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)}$$