1. (20 pts)

a) Calculate $\lim\limits_{x \to 0} \dfrac{x^2 + 3x - \sin^2 x}{2x^2}$

b) Calculate $\lim\limits_{x \to \infty} \dfrac{x^2 + 3x - \sin^2 x}{2x^2+1}$

c) Using implicit differentiation, find the equation of the tangent line to the cissoid of Diocles $y^2(2-x) = x^3$ at $(1,1)$.

Solution:

a) $$\lim_{x \to 0} \frac{x^2+3x-\sin^2 x}{2x^2} = \lim_{x \to 0} \left(\frac{1}{2} + \frac{3}{2x} - \frac{1}{2}\frac{\sin^2 x}{x^2} \right)$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{3}{2} \lim_{x \to 0} \frac{1}{x}$$

$$= \frac{3}{2} \lim_{x \to 0} \frac{1}{x}$$

$$= \boxed{\text{DNE}}$$

$\lim\limits_{x \to 0^-} = -\infty$ & $\lim\limits_{x \to 0^+} = +\infty$

Limits don't agree $\Rightarrow$ DNE

b) $$\lim_{x \to \infty} \frac{x^2 + 3x - \sin^2 x}{2x^2+1} = \lim_{x \to \infty} \frac{x^2\left(1 + \frac{3}{x} - \frac{\sin^2 x}{x^2}\right)}{x^2\left(2 + \frac{1}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{3}{x} - \frac{\sin^2 x}{x^2}}{2 + \frac{1}{x^2}}$$

$$= \frac{1+0+0}{2+0}$$

$$\boxed{= \frac{1}{2}}$$

c) $$y^2(2-x) = x^3$$

$$\Rightarrow 2y\frac{dy}{dx}(2-x) + y^2(-1) = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2+y^2}{2y(2-x)}$$

At $(x,y) = (1,1)$;

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{3+1}{2(2\cdot 1)} = \frac{4}{2} = 2$$

Thus the tangent line is:

$$y_t(x) = y(1) + y'(1)(x-1)$$

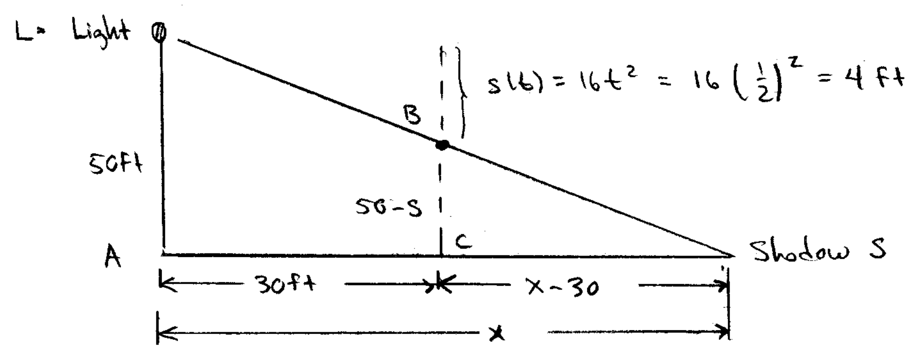
$$= 1 + 2(x-1)$$

$$\Rightarrow \boxed{y_t(x) = 2x - 1}$$

2. (20 pts)

A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30ft away from the light. How fast is the shadow of the ball moving along the ground 0.5 seconds later? (Assume the ball falls a distance $s = 16t^2$ in t seconds).

Solution:



We want $\frac{dx}{dt}$. Now, the triangle ASL is similiar to the triangle CSB. Hence:

$$\frac{50}{x} = \frac{50 - s(t)}{x - 30}$$

$$\Rightarrow 50(x-30) = x(50-s)$$

$$\Rightarrow -150 = -sx \qquad (1)$$

$$\Rightarrow \frac{d}{dt}(-150) = \frac{d}{dt}(-sx)$$

$$\Rightarrow 0 = -x\frac{ds}{dt} - s\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = -\frac{x}{s}\frac{ds}{dt}$$

$$= -\frac{150}{s^2}\frac{ds}{dt} \qquad \text{using (1)}$$

$$= -\frac{150}{s^2}\frac{d}{dt}(16t^2)$$

$$= -\frac{150}{s^2}32t$$

$$= -\frac{150}{(16t^2)^2}(32t)$$

When $t = \frac{1}{2}$:

$$\frac{dx}{dt} = -\frac{150}{(4)^2}(16)$$

$$\Rightarrow \boxed{\frac{dx}{dt} = -150\frac{ft}{sec}}$$

3. (20 pts) Define $f(x) = \cos(\sqrt[3]{x+1}\,\pi)$

a) Calculate the linearization $L(x)$ of $f(x)$ about $x=0$

b) Use $L(x)$ from (a) to estimate $\cos(\sqrt[3]{1.1}\,\pi)$.

Solution:

a) $$L(x) = f(x_0) + f'(x_0)(x-x_0)$$

Hence: $$f(x) = \cos\left[(x+1)^{1/3}\pi\right]$$

$$f'(x) = -\sin\left[(x+1)^{1/3}\pi\right]\,\frac{1}{3}\,(x+1)^{-2/3}\pi$$

$$= -\frac{\pi\sin(\sqrt[3]{(x+1)}\,\pi)}{3(x+1)^{2/3}}$$

At $x=0$; $$f(0) = \cos\pi = -1$$

$$f'(0) = -\frac{\pi\sin\pi}{3} = 0$$

So

$$\boxed{L(x) = -1}$$

b) $$\boxed{\cos(\sqrt[3]{1.1}\,\pi) \approx -1}$$

4. (20 points)

Consider the function $f(x) = x - \dfrac{1}{x-1}$ for $x \neq 1$.

a) Find the x and y coordinates of all local maxima and minima. Identify which are maxima and which are minima. Justify your answers.

b) Determine any horizontal or vertical asymptotes the graph of f might have.

c) Graph f using the information from (a) and (b). In your graph show and label all maxima, minima, and asymptotes.

Solution:

a)
$$f(x) = x - \frac{1}{x-1}$$

$$\Rightarrow f'(x) = 1 + \frac{1}{(x-1)^2}$$

Note that $f'(x) > 0$ for all $x \neq 1$. Hence it is always increasing and thus has no critical points. Hence:

$$\boxed{\text{No local or absolute extrema}}$$

b)

$$\lim_{x \to 1^-} f(x) = +\infty$$

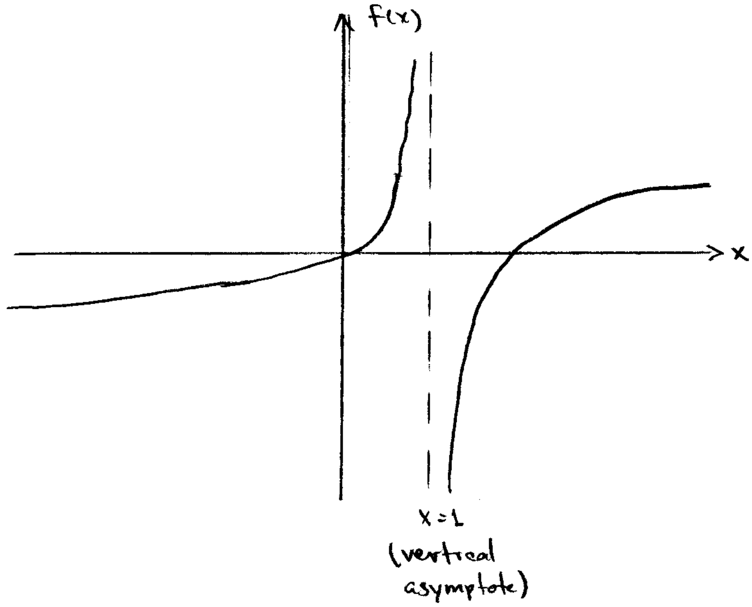
$$\lim_{x \to 1^+} f(x) = -\infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

$$\lim_{x \to +\infty} f(x) = +\infty$$

$\Rightarrow$ $\boxed{\begin{array}{l}\text{Vertical asymptote at } x = 1 \\ \text{No horizontal asymptotes}\end{array}}$

c)



5. (20 points)

(a) Show that at some instant during a 2-hour automobile trip, the speedometer reading will equal the average speed of the trip.

(b) To make a box, you use a square piece of cardboard with sidelengths L = 10 in. Out of all four corners of the piece you cut identical squares of sidelengths x inches, then you fold up the edges. How large does x have to be for the box to have the largest volume?

Solution:

a) Assume we leave at $t = 0$ and arrive at $t = 2$. The position $s(t)$ will be continuous on $[0, 2]$ and differentiable on $(0, 2)$. From the mean value theorem, there will be some $t_a \in (0, 2)$ where

$$\frac{s(2) - s(0)}{2 - 0} = v(t_a)$$

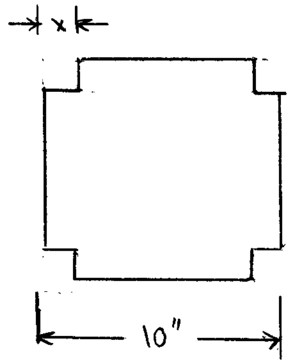
where $v(t) = \frac{ds}{dt}$ is the velocity. The left-hand side is the average velocity. Take the absolute value of both sides:

$$\Rightarrow \left|\frac{s(2)-s(1)}{2}\right| = |v(t_a)|$$

average speed

speedometer reading at some time $t_a \in (0,2)$

b)



Once folded the base will have length $10-2x$ and the box will have height x. Thus the volume will be

$$V = (10-2x)^2 x$$

Note that $0 < x < 5$. There are no endpoints for x, hence the extrema of the volume must be at a critical point on $(0,5)$. Thus

$$\begin{aligned} \frac{dV}{dx} &= \frac{d}{dx}(10-2x)^2 x \\ &= 2x(10-2x)(-2) + (10-2x)^2 \\ &= (10-2x)(-4x+10-2x) \\ &= (10-2x)(10-6x) \end{aligned}$$

Then $\frac{dV}{dx} = 0 \Rightarrow x = 5$ or $x = \frac{5}{3}$. $\Rightarrow$ $\boxed{x = \frac{5}{3} \text{ inches}}$

outside range of x