APPM 1350: Exam III Solutions

1) a) Estimate the integral $\int_1^2 \frac{1}{x}\,dx$ using $n=4$ rectangles and left-hand endpoints.

Soln: The $n=4$ partition of $[1,2]$ will be:

$$x_0 < x_1 < x_2 < x_3 < x_4 \qquad h = \Delta x = \frac{2-1}{4} = \frac{1}{4}$$

where
$x_0 = 1$
$x_1 = 5/4$
$x_2 = \frac{3}{2}$
$x_3 = 7/4$
$x_4 = 2$

The corresponding y values will be:

$y_0 = 1$
$y_1 = 4/5$
$y_2 = 2/3$
$y_3 = 4/7$
$y_4 = 1/2$

The rectangle approximation using left-hand endpoints will be:

$$\int_1^2 \frac{dx}{x} \approx \sum_{k=1}^{4} f(x_{k-1})\,h$$

$$= \frac{1}{4}\left(y_0 + y_1 + y_2 + y_3\right)$$

$$= \boxed{\frac{1}{4}\left(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) = \frac{319}{420}}$$

b) Estimate the same integral using the trapezoidal sum for $n=4$

Soln:

Using the trapezoidal rule;

$$\int_1^2 \frac{dx}{x} \approx \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

$$= \boxed{\frac{1}{8}\left(1 + \frac{8}{5} + \frac{4}{3} + \frac{8}{7} + \frac{1}{2}\right) = \frac{1171}{1680}}$$

c) How large would you have to make n to be sure that the corresponding trapezoidal estimate is within 0.01 of the real value of the integral?

Soln:

We want $|E_T| \leq \frac{b-a}{12} h^2 M \leq \epsilon$ where $\epsilon = 0.01$, $h = \frac{b-a}{n}$
and M is given by

$$M \geq \max_{x \in [a,b]} |f''(x)|$$

In our case ; $f(x) = \frac{1}{x}$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

So on $x \in [1,2]$; $\max|f''(x)| = |f''(1)| = 2$. Thus:

$$\frac{2-1}{12}\left(\frac{2-1}{n}\right)^2 2 \leq 0.01$$

$$\Rightarrow n^2 \geq \frac{1}{6\cdot(0.01)} = \frac{100}{6} = \frac{50}{3}$$

$$\Rightarrow n^2 \geq \frac{50}{3} \Rightarrow \boxed{n \geq \sqrt{\frac{50}{3}}} = 4.08. \text{ Round up} \Rightarrow \boxed{n \geq 5}$$

2) Find the total area of the two regions enclosed between the graph of the function $y = 3x - x^2$ on the interval $[0,4]$ and the x-axis

Soln:

$$\text{Area} = \int_0^4 |3x - x^2|\,dx$$

Now $3x - x^2 = x(3-x)$ has roots at $x=0$ and $x=3$. On $[0,3]$ the function is positive whereas on $[3,4]$ it is negative.
So:

$$|3x-x^2| = \begin{cases} 3x - x^2 & x \in [0,3] \\ -(3x-x^2) & x \in [3,4] \end{cases}$$

and hence

$$\text{Area} = \int_0^4 |3x-x^2|\,dx$$

$$= \int_0^3 |3x-x^2|\,dx + \int_3^4 |3x-x^2|\,dx$$

$$= \int_0^3 (3x-x^2)\,dx + \int_3^4 -(3x-x^2)\,dx$$

$$= \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^3 - \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Bigg|_3^4$$

$$= \left[\frac{27}{2} - \frac{27}{3}\right] - \left[\left(\frac{48}{2} - \frac{64}{3}\right) - \left(\frac{27}{2} - \frac{27}{3}\right)\right]$$

$$= 2\left(\frac{27}{2} - \frac{27}{3}\right) - \left(\frac{48}{2} - \frac{64}{3}\right)$$

$$= \frac{19}{3}$$

3) Calculate the following integrals:

a) $$\int \frac{\sqrt{v}+1}{\sqrt[3]{v}}\,dv = \int \frac{v^{1/2}+1}{v^{1/3}}\,dv$$

$$= \int \left(v^{1/6} + v^{-1/3}\right) dv$$

$$= \frac{v^{7/6}}{(7/6)} + \frac{v^{2/3}}{(2/3)} + c$$

$$= \boxed{\frac{6}{7}v^{7/6} + \frac{3}{2}v^{2/3} + c}$$

b) $$\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx = 2\int \frac{du}{u^2}$$

Let $u = 1+\sqrt{x}$

$du = \dfrac{dx}{2\sqrt{x}}$

$$= -\frac{2}{u} + c$$

$$= \boxed{-\frac{2}{1+\sqrt{x}} + c}$$

c) $$\int_0^1 r\sqrt{1-r^2}\,dr = -\frac{1}{2}\int_1^0 u^{1/2}\,du$$

Let $u = 1-r^2$

$du = -2r\,dr$

$r=0 \Rightarrow u=1$

$r=1 \Rightarrow u=0$

$$= \frac{1}{2}\int_0^1 u^{1/2}\,du$$

$$= \frac{u^{3/2}}{2(3/2)}\Bigg|_0^1$$

$$= \frac{1}{3}u^{3/2}\Bigg|_0^1$$

$$= \boxed{\frac{1}{3}}$$

4. a) State the Fundamental Theorem of Calculus (both parts)

a) Let $f(x)$ be continuous on $[a,b]$ and define $F(x) = \int_a^x f(t)\,dt$.
Then $F'(x)$ exists for all $x \in [a,b]$ and

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

b) Let $f(x)$ be continuous on $[a,b]$ and let $F(x)$ be any antiderivative of $f(x)$ on $[a,b]$. Then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

b) Calculate the derivative of the function $h(x) = \int_0^{x^2} \cos\sqrt{\theta}\,d\theta$

Soln:

$$\frac{d}{dx}\int_0^{x^2} \cos\sqrt{\theta}\,d\theta = \cos\sqrt{x^2}\,\frac{d}{dx}x^2$$

$$= 2x\cos\sqrt{x^2}$$

$$\boxed{\begin{aligned} &= 2x\cos|x| \\ &= 2x\cos x \end{aligned}}$$

since $\cos x$ is an even function

c) Find the critical points of the function h given in (b).

Soln: The critical points are where $h'(x) = 0$. So

$$h'(x) = 2x\cos x = 0$$

$$\Rightarrow \boxed{x = 0, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots}$$

5) a) Find the average value of the function $f(x) = x^2 - 1$ on $[0, \sqrt{3})$

Soln:
$$av(f) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

$$\Rightarrow av(f) = \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} (x^2-1)\,dx$$

$$= \frac{1}{\sqrt{3}} \left(\frac{x^3}{3} - x \right) \Big|_0^{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \left(\sqrt{3} - \sqrt{3} \right)$$

$$= \boxed{0}$$

b) The Mean Value Theorem states that there is a value $x = c$ such that $f(c)$ equals the average value of f. Find this value of c.

Soln:
$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx = av(f)$$

In our case $f(x) = x^2 - 1$, so plugging in $x = c$ and $av(f) = 0$:

$$c^2 - 1 = 0$$

$$\Rightarrow c^2 = 1$$

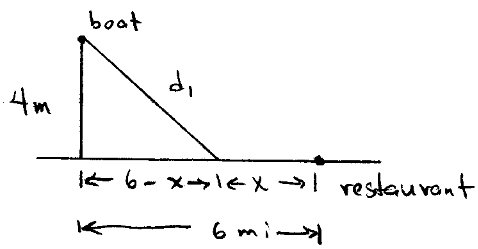
$$\Rightarrow c = \pm 1$$

However our interval was $[0, \sqrt{3}]$, so only $\boxed{c = 1}$ is in this interval.

Extra-credit

A boat is 4 miles from the nearest point on a straight shore line which is 6 miles from a shoreside restaurant. A woman plans to row to a point on shore and then walk to the restaurant. If she can walk at 3 mph, at what speed must she be able to row so that the quickest way to get to the restaurant is instead to row there directly?

Soln:



Let v be the rowing speed. Assume she rows to a point x miles from the restaurant as shown. She will then be rowing a distance d_1 where

$$d_1 = \sqrt{(6-x)^2 + (4)^2}$$

She will be walking a distance $d_2 = x$. The time T_1 spent rowing will be

$$T_1 = \frac{d_1}{v} = \frac{\sqrt{(6-x)^2+16}}{v}$$

whereas the time T_2 spent walking will be

$$T_2 = \frac{d_2}{3} = \frac{x}{3}$$

Hence the total time to get to the restaurant will be:

$$T(x) = T_1(x) + T_2(x) = \frac{\sqrt{(6-x)^2+16}}{v} + \frac{x}{3}$$

depending on how far x from the restaurant she lands her boat. To find the value of x that will minimize the time to get to the restaurant, we compute $dT/dx = 0$. Hence

$$\frac{dT}{dx} = \frac{-2(6-x)}{2v\sqrt{(6-x)^2+16}} + \frac{1}{3} = \frac{x-6}{v\sqrt{(6-x)^2+16}} + \frac{1}{3} = 0$$

So now let $x = 0$ (she rows directly to the restaurant) and find v so that this is the minimal time. Thus $v = \frac{3(6-x)}{\sqrt{(6-x)^2+16}}$ with $x=0$. Hence $\boxed{v = \frac{18}{\sqrt{52}}}$