APPM 1350: Section 1.2 Supplemental : The Sandwich Theorem

Thm (The Sandwich Theorem): Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x=c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

Then $\lim_{x\to c} f(x) = L$

Example

(#44, p.66) If $2-x^2 \le f(x) \le 2\cos x$ for all x, Find $\lim_{x\to 0} f(x)$

Solution:

Let $g(x) = 2-x^2$

$h(x) = 2\cos x$

Then:
$$\lim_{x\to 0} g(x) = \lim_{x\to 0} (2-x^2)$$
$$= 2$$

and
$$\lim_{x\to 0} h(x) = \lim_{x\to 0} 2\cos x$$
$$= 2$$

So $\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = 2$ and $g(x) \le f(x) \le h(x) \Rightarrow \lim_{x\to 0} f(x) = 2$

Example
(#46, p. 66)

Suppose $\frac{1}{2} - \frac{x^2}{24} < \frac{1-\cos x}{x^2} < \frac{1}{2}$

holds for $|x| << 1$. What is the limit $\lim_{x\to 0} \frac{1-\cos x}{x^2}$?

Solution:

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1-\cos x}{x^2} < \frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0} \left(\frac{1}{2} - \frac{x^2}{24}\right) \leq \lim_{x\to 0} \left(\frac{1-\cos x}{x^2}\right) \leq \lim_{x\to 0} \left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{1}{2} \leq \lim_{x\to 0} \left(\frac{1-\cos x}{x^2}\right) \leq \frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$