APPM 1350 – Exam I
Solutions

1. (a) domain $(f) = \mathbb{R}$ = all real numbers

domain $(g) = \{x \geq 0\}$ = all positive or zero real numbers

domain (h) = all values of x where $\cos(x/2) \neq 0$

$= \{\frac{x}{2} \neq 2k\pi \pm \frac{\pi}{2}\} = \{x \neq 4k\pi \pm \pi\}$

$= \{x \neq$ any odd multiple of $\pi\}$

(b) $(f \circ g)(x) = f(\sqrt{x}) = \sqrt{x} - 7$

(c) domain $(f \circ g) = \{x \geq 0\}$

range $(f \circ g) = \{y \geq -7\}$ (since $\sqrt{x} \geq 0 \Rightarrow y = \sqrt{x} - 7 \geq 0 - 7 = -7$)

2. (a) The derivative of f at $x = a$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(b) $f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{f(1+h) - (-1)}{h}$

simplify first: $f(x) = \frac{x-2}{x} = \frac{x}{x} - \frac{2}{x} = 1 - \frac{2}{x} \Rightarrow f(1+h) = 1 - \frac{2}{1+h}$

$$f'(1) = \lim_{h \to 0} \frac{1 - \frac{2}{1+h} - (-1)}{h} = \lim_{h \to 0} \frac{2 - \frac{2}{1+h}}{h} = \lim_{h \to 0} \frac{\frac{\cancel{2} + 2h - \cancel{2}}{1+h}}{h} =$$

$$= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}(1+h)} = \lim_{h \to 0} \frac{2}{1+h} = \boxed{2}$$

(c) Use the point-slope formula for slope $m = 2$ and point $(1,-1)$:

$$\frac{y-(-1)}{x-1} = 2 \Rightarrow \frac{y+2}{x-1} = 2 \Rightarrow y+2 = 2(x-1)$$

$$y = 2(x-1)-2 \Rightarrow \boxed{y = 2x-4}$$

3. (a) $$\lim_{t\to 1} \frac{t^2+t-2}{t^2-1} = \lim_{t\to 1} \frac{\cancel{(t-1)}(t+2)}{\cancel{(t-1)}(t+1)} = \boxed{\frac{3}{2}}$$

(b) $$\lim_{s\to -1} \frac{\sqrt{s^2+8}-3}{s+1} =$$ (multiply by the conjugate $\sqrt{s^2+8}+3$)

$$\lim_{s\to -1} \frac{s^2+8-9}{(s+1)(\sqrt{s^2+8}+3)} = \lim_{s\to -1} \frac{s^2-1}{(s+1)(\sqrt{s^2+8}+3)} =$$

$$\lim_{s\to -1} \frac{\cancel{(s+1)}(s-1)}{\cancel{(s+1)}(\sqrt{s^2+8}+3)} = \frac{-2}{\sqrt{(-1)^2+8}+3} = \frac{-2}{\sqrt{9}+3} = -\frac{2}{6} = \boxed{-\frac{1}{3}}$$

(c) $$\lim_{h\to 0} \frac{h+2}{h^4+h^2} = \boxed{+\infty,}$$ since $\lim_{h\to 0}(h+2) = 2$ and $\lim_{h\to 0}(h^4+h^2) = 0^+$

4. (a) $$y = \frac{2x+5}{3x-2}$$

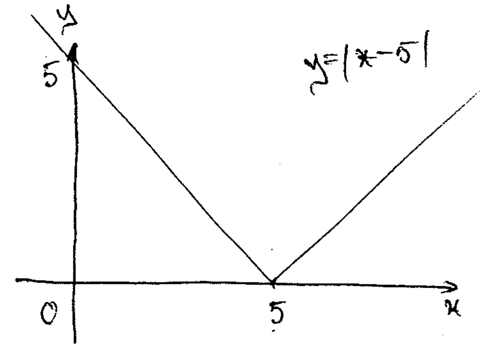
The function is differentiable at any $x \neq \frac{2}{3}$, ~~where~~ (since at $x = \frac{2}{3}$ it is not defined).

For any $x \neq \frac{2}{3}$:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x+5}{3x-2}\right) \overset{\text{quotient}}{\underset{\text{rule}}{=}} \frac{2(3x-2)-3(2x+5)}{(3x-2)^2} =$$

$$= \frac{\cancel{6x}-4-\cancel{6x}-15}{(3x-2)^2} = \boxed{-\frac{19}{(3x-2)^2}}$$

(b) $y = |x-5| = \begin{cases} x-5, & \text{if } x \geq 5 \\ -(x-5), & \text{if } x < 5 \end{cases}$



The function is not differentiable at $x=5$
(the tangent line does not exist at $(5,0)$)

For $x<5$: $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(-(x-5)\right) = \boxed{-1}$

For $x \geq 5$: $\dfrac{dy}{dx} = \dfrac{d}{dx}(x-5) = \boxed{1}$

5. (a) A function f is continuous at $x=c$ if

(i) $\lim\limits_{x\to c} f(x)$ exists (as a finite number)

(ii) the value $f(c)$ exists (i.e., c is in the domain of f)

(iii) $\lim\limits_{x\to c} f(x) = f(c)$

(b) The function f is continuous everywhere except at $x=3$, for any value of a.
To make it also continuous at $x=3$, we need:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$$

$$\lim_{x\to 3^-}(x^2-1) = \lim_{x\to 3^+}(2ax) = 2\cdot a\cdot 3 \Rightarrow 9-1 = 6a \Rightarrow 6a = 8 \Rightarrow a = \frac{8}{6} \Rightarrow$$

$$\boxed{a = \frac{4}{3}}$$

6. (a) Average velocity between $t=0$ and $t=1$:

$$\bar{v} = \frac{s(1)-s(0)}{1-0} = \frac{(100-16)-100}{1-0} = \frac{-16}{1} = -16 \text{ ft/sec}$$

(b) Instantaneous velocity at $t=1$:

$v(1) = s'(1) = -16 \cdot 2 \cdot 1 = -32$ ft/sec

Instantaneous speed at $t=1$:

$|v(1)| = 32$ ft/sec

Acceleration at $t=1$:

$a(1) = v'(1) = s''(1) = -32$ ft/sec^2

(c) At $t=1$, the object is moving downwards ($v(1)<0$), so negative acceleration means that it's speeding up. (It's what we would expect from an object falling off a cliff!!)

Extra-credit: we want to find the points where $\frac{dy}{dx} = 0$.

We will need $\frac{d}{dx}\left(\sqrt{1-x^2}\right)$, so we calculate it below:

$$\frac{d}{dx}\left(\sqrt{1-x^2}\right) = \lim_{h \to 0} \frac{\sqrt{1-(x+h)^2} - \sqrt{1-x^2}}{h} = \lim_{h \to 0} \frac{\cancel{1}-(x+h)^2-\cancel{1}+x^2}{h\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} =$$

$$= \lim_{h \to 0} \frac{-\cancel{h}(2x+h)}{\cancel{h}\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}.$$

$$\frac{d}{dx}\left(|x| \pm \sqrt{1-x^2}\right) = \pm 1 \pm \frac{x}{\sqrt{1-x^2}}$$

this can $=0$ if $\frac{x}{\sqrt{1-x^2}} = 1 \Rightarrow x^2 = 1-x^2 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$

$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$. The corresponding points are

$\left(\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$