Unit Exam 2 Solutions

1a) $\lim_{x\to\infty} \frac{2\sqrt{x}+x^{-1}}{3x-7} = \lim_{x\to\infty} \frac{\frac{2}{\sqrt{x}}+\frac{1}{x^2}}{3-\frac{7}{x}} = \boxed{0}$ (divide each term by x) (degree of denom > degree of num)

1b) $\lim_{x\to-\infty} \frac{1-x+x^2}{2+x-2x^2} = \frac{\frac{1}{x^2}-\frac{1}{x}+1}{\frac{2}{x^2}+\frac{1}{x}-2} = \boxed{-\frac{1}{2}}$ (divide each term by x^2) (num & denom of same degree)

1c) $\lim_{t\to 0} \frac{2t}{\tan t} = \lim_{t\to 0} \frac{2t}{\frac{\sin t}{\cos t}} = \lim_{t\to 0} \frac{t}{\sin t} \cdot 2\cos t = \lim_{t\to 0} 2\cos t = \boxed{2}$ $\left(\lim_{t\to 0} \frac{\sin t}{t} = 1\right)$

2) The tangent line will be $\parallel$ to the x-axis when $\frac{dy}{dx}=0$.

$x^2+xy+y^2=7$

$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$

$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x-y$

$\frac{dy}{dx} = \frac{-2x-y}{x+2y}$

$\frac{dy}{dx}=0$ when $-2x-y=0$
or $y=-2x$

Substituting $y=-2x$ into orig eq:

$x^2+x(-2x)+(-2x)^2=7$

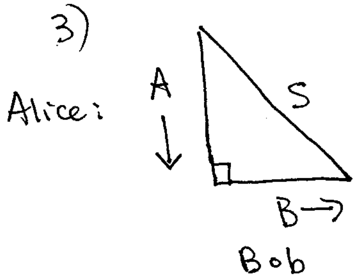
$x^2-2x^2+4x^2=7$

$x^2=\frac{7}{3}$

$x=\pm\sqrt{\frac{7}{3}}$

Since $y=-2x$, the 2 points where the tangent line is $\parallel$ to the x-axis are $\boxed{\left(\sqrt{\frac{7}{3}}, -2\sqrt{\frac{7}{3}}\right) \text{ and } \left(-\sqrt{\frac{7}{3}}, 2\sqrt{\frac{7}{3}}\right)}$.

3)



Given: $\frac{dA}{dt} = -2$ m/sec (the rate is negative because A is decreasing)

$\frac{dB}{dt} = 1.5$ m/sec (the rate is positive because B is increasing)

Find: $\frac{dS}{dt}$ when $A=12$m, $B=9$m.

$A^2+B^2=S^2$

$S=\sqrt{A^2+B^2}=\sqrt{12^2+9^2}=15$ m

$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2S\frac{dS}{dt}$

$2(12)(-2)+2(9)(1.5) = 2(15)\frac{dS}{dt}$

$-48+27 = 30\frac{dS}{dt}$

$\frac{dS}{dt} = \frac{-21}{30} = \boxed{\frac{-7}{10} \text{ m/sec}}$

4) We rewrite the equation as $x - \frac{1}{3-x^2} = 0$ and search for a root of $f(x) = x - \frac{1}{3-x^2} = 0$.

a) Since f is continuous on $(0,1)$ and $f(0) = -\frac{1}{3}$ and $f(1) = \frac{1}{2}$, by the Intermediate Value Theorem, $f(x) = 0$ must have at least one solution in $(0,1)$. (Since f changes sign, it must cross the x-axis in the interval.)

b) $f'(x) = 1 - \frac{2x}{(3-x^2)^2}$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

let $x_0 = 0$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$
0	0	$-1/3$	1	$0 + 1/3 = 1/3$
1	$\boxed{1/3}$			

5) $f(x) = \frac{x^2+x+1}{x+1}$ for $x \neq -1$ or $f(x) = x + \frac{1}{x+1}$ for $x \neq -1$

$f'(x) = 1 - \frac{1}{(x+1)^2}$

a) $f'(x) = \frac{(x+1)(2x+1) - (x^2+x+1)}{(x+1)^2}$

$f'(x) = \frac{x^2+2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$

Critical points: f' is undefined for $x = -1$
$f' = 0$ for $x = 0$ or $x = -2$

Rise and fall:

$$\begin{array}{ccccccc} + & & - & & - & & + \\ & -2 & & -1 & & 0 & \\ & \text{local max} & & & & \text{local min} & \end{array}$$

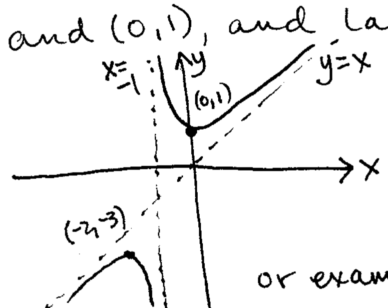
$x = -2$, $y = -3$ is a local max because y' changes from pos to neg

$x = 0$, $y = +1$ is a local min because y' changes from neg to pos

b) No horizontal asymptotes exist because
$\lim_{x \to \infty} x + \frac{1}{x+1} = \infty$ and $\lim_{x \to -\infty} x + \frac{1}{x+1} = -\infty$

A vertical asymptote at $x = -1$ exists because
$\lim_{x \to -1^+} x + \frac{1}{x+1} = \infty$ and $\lim_{x \to -1^-} x + \frac{1}{x+1} = -\infty$

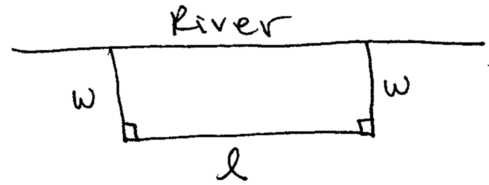
c) We plot the local extrema $(-2,-3)$ and $(0,1)$, and label the vertical asymptote $x = -1$.
Note that $y = x$ is an oblique asymptote because the x term dominates when $x \to \pm\infty$ in $x + \frac{1}{x+1}$.
Alternatively, you can plot specific pts or examine concavity.



6a) If $f(x)$ is continuous on $[a,b]$ and differentiable on the interior (a,b), then there is at least one point c in (a,b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

6b)



We wish to maximize the area of the rectangle given $P = 2w + l = 800\text{ m}$.

Area $A = lw$

We substitute $l = 800 - 2w$.

$A = (800-2w)w$

$A = 800w - 2w^2$, where $0 \le w \le 400$.

We examine values where $\frac{dA}{dw} = 0$.

$$\frac{dA}{dw} = 800 - 4w \quad , \quad \frac{d^2A}{dw^2} = -4$$

$800 - 4w = 0$

$w = 200\text{m}$

Since $\frac{d^2A}{dw^2} = -4$, the graph of A is concave down and $w = 200\text{m}$, $l = 400\text{m}$ is a local maximum.

Checking the endpoints yields $A(0) = A(400) = 0$.

Therefore the maximum area occurs when $w = 200\text{m}$, $l = 400\text{m}$, and

$$A = \boxed{80{,}000\text{ m}^2}.$$

5c) Larger graph: