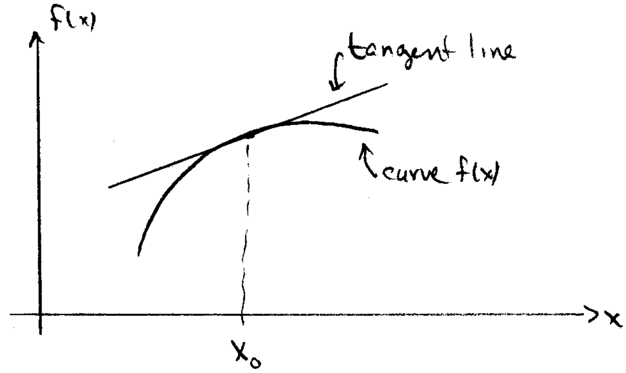
APPM 1350: Section 1.6: Tangent Lines

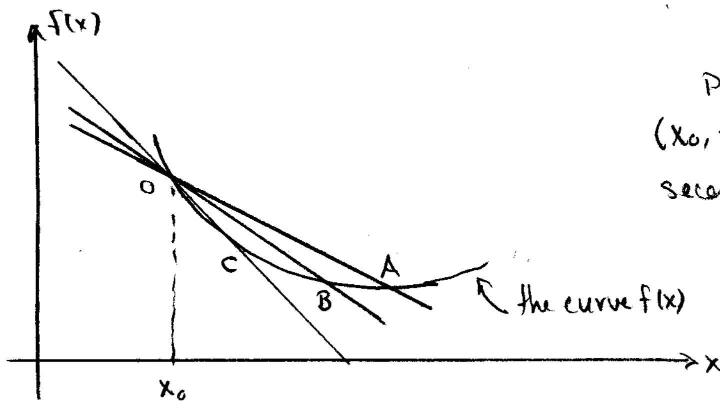
Tangents to Curves

Given some function $f(x)$ on some interval, we will often talk about the tangent to the function (or curve) at some point x. What do we mean by tangent?



The tangent to $f(x)$ at x_0 is a line that touches the curve at x_0. Imagine the curve was a roadway and you are driving on it at night with your headlights on. The direction that your headlights point is the direction of the straight line tangent to the roadway (curve) at that point.

A more mathematical definition of the tangent line is given as a limit of secant lines.



Consider some point O located at $(x_0, f(x_0))$. Look at the secant lines OA, OB, and OC. As noted in section 1.1, the slope of the secant lines approaches a limit as the endpoint A, B, or C gets closer to O. In this limit, the tangent line is the line whose slope is equal to that limit. So...

Def The <u>slope of the curve</u> $y=f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

(provided the limit exists). The <u>tangent line</u> to the curve at P is the line through P with this slope.

The <u>equation of the tangent line at $x=x_0$</u> is given by

$$y(x) = y_0 + m(x-x_0)$$

where $y_0 = f(x_0)$ and

$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

provided the limit exists

<u>Example</u>

Find the slope and equation of the line tangent to $f(x) = (x-2)^2$ at $x=2$.

Solution:

$$m = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h-2)^2-(2-2)^2}{h}$$

$$= \lim_{h\to 0} \frac{h^2}{h}$$

$$= 0$$

So the slope of the tangent line at $x=0$ is zero, As for the equation of the tangent line;

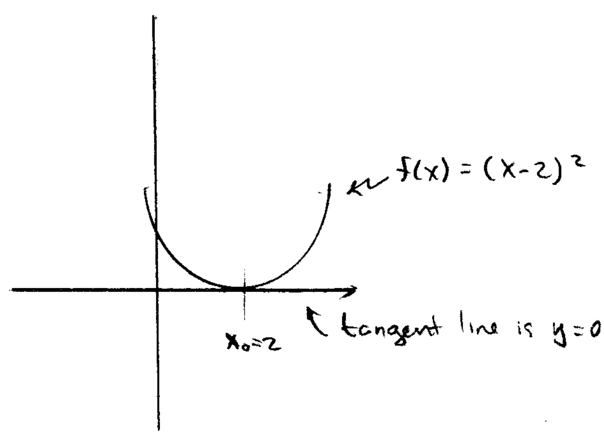
$$y_0 = f(2) = (2-2)^2 = 0$$

$$\Rightarrow \quad y(x) = y_0 + m(x - x_0)$$

$$= 0 + 0(x-2)$$

$$= 0$$

So $y(x)=0$ is the equation of the line tangent to x^2 at $x=0$:



Example

Find the slope and equation of the line tangent to $f(x) = (x-2)^2$ at $x = -4$.

Solution:

Same idea as before:

$$m = \lim_{h \to 0} \frac{f(-4+h) - f(-4)}{h}$$

$$= \lim_{h \to 0} \frac{(-4+h-2)^2 - (-4-2)^2}{h}$$

$$= \lim_{h \to 0} \frac{(-6+h)^2 - (-6)^2}{h}$$

$$= \lim_{h \to 0} \frac{36 - 12h + h^2 - 36}{h}$$

$$= \lim_{h \to 0} \frac{h(-12+h)}{h}$$

$$= \lim_{h \to 0} (-12+h)$$

$$= -12$$

The equation of the line tangent to x^2 at $x=-4$ is:

$$y = y_0 + m(x - x_0)$$

$$= f(-4) + (-12)(x-(-4))$$

$$= (-4)^2 - 12(x+4)$$

$$= 16 - 12(x+4)$$

You could simplify this if you wish to:

$$y = -32 - 12x$$

Example

Find the equation of the line tangent to $f(x) = x^3$ for arbitrary $x_0 \in \mathbb{R}$

Solution:

$$m = \lim_{h \to 0} \frac{(x_0+h)^3 - x_0^3}{h}$$

$$= \lim_{h \to 0} \frac{x_0^3 + 3x_0^2 h + 3x_0 h^2 + h^3 - x_0^3}{h}$$

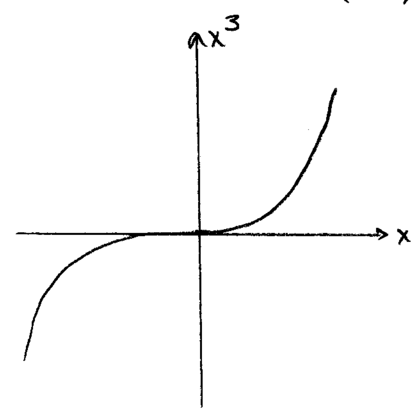
$$= \lim_{h \to 0} \frac{3x_0^2 h + 3x_0 h^2 + h^3}{h}$$

$$= \lim_{h \to 0} (3x_0^2 + 3x_0 h + h^2)$$

$$= 3x_0^2$$

1
1 2 1
1 3 3 1

Pascal's triangle. Neat way to expand $(x+a)^n$



The tangent line is given by:

$$y = y_0 + m(x - x_0)$$

$$= x_0^3 + 3x_0^2(x - x_0)$$

$$= x_0^3 + (3x_0^2)x - 3x_0^3$$

$$= -2x_0^3 + (3x_0^2)x$$

Example

Find the equation of the straight line having slope 27 that is tangent to the curve $y = x^3$.

Solution:

From the previous example;

$$m = 3x_0^2$$

$$\Rightarrow 27 = 3x_0^2$$

$$\Rightarrow x_0 = 3$$

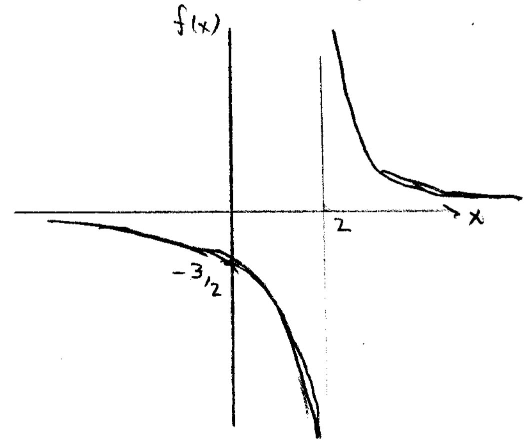
and the equation of the tangent line at this point is:

$$y = -2x_0^3 + 3x_0^2 x$$

$$= -2(3)^3 + 3(3)^2 x$$

$$= -54 + 27x$$

<u>Example</u>



Find the equation for the line tangent to

$$f(x) = \frac{x+3}{x-2}$$

at an arbitrary point $x_0 \neq 2$.

Solution:

$$m = \lim_{h\to 0} \frac{\dfrac{(x_0+h)+3}{(x_0+h)-2} - \dfrac{x_0+3}{x_0-2}}{h} \qquad (x \neq 2)$$

$$= \lim_{h\to 0} \frac{[(x_0+h)+3][x_0-2] - [(x_0+h)-2][x_0+3]}{h[(x_0+h)-2][x_0-2]}$$

$$= \lim_{h\to 0} \frac{[\cancel{x_0^2} - \cancel{2x_0} + \cancel{hx_0} - 2h + \cancel{3x_0} - \cancel{6}] - [\cancel{x_0^2} + \cancel{3x_0} + \cancel{hx_0} + 3h - \cancel{2x_0} - \cancel{6}]}{h[(x_0+h)-2][x_0-2]}$$

$$= \lim_{h\to 0} \frac{-5\cancel{h}}{\cancel{h}[(x_0+h)-2][x_0-2]}$$

$$= -\frac{5}{(x_0-2)^2}$$

Hence:

$$y = y_0 + m(x-x_0)^2$$

$$= \frac{x_0+3}{x_0-2} - \frac{5}{(x_0-2)^2}(x-x_0) \longleftarrow \text{eq. of tangent line to } \frac{x-3}{x-2} \text{ at } x = x_0$$

Vertical Tangents

Def The curve $y = f(x)$ has a vertical tangent at $x = x_0$ if

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \pm\infty$$

Example

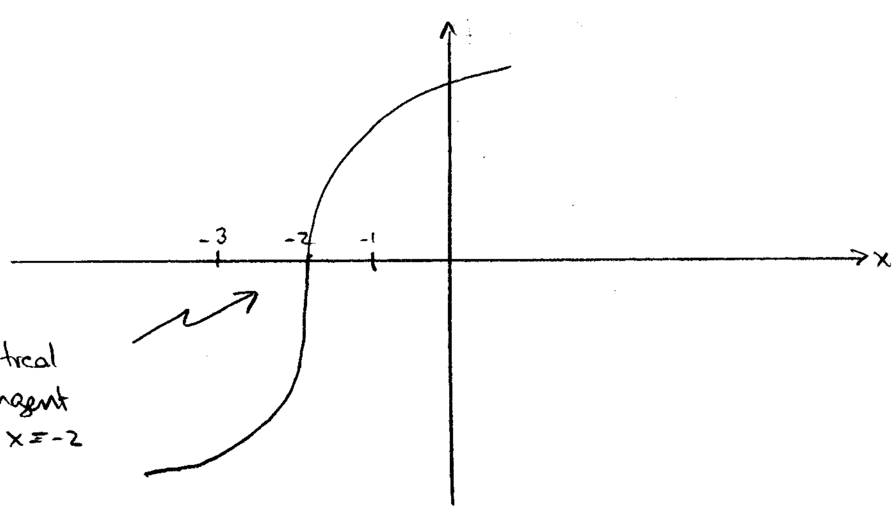
Consider $f(x) = (x+2)^{1/5}$. At $x = -2$ we have:

$$\lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0} \frac{h^{1/5}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h^{4/5}}$$

$$= \lim_{h \to 0} h^{-4/5}$$

$$= +\infty$$



vertical tangent at $x = -2$

When Tangent Lines Don't Exist

If the difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

has different left- and right-handed limits at some point $x=x_0$ as $h \to 0^{\pm}$, then there is no line that is tangent to $f(x)$ at $x=x_0$.

Example

$$\text{Let } f(x) = |x| = \begin{cases} -x & x<0 \\ x & \geq 0 \end{cases}$$

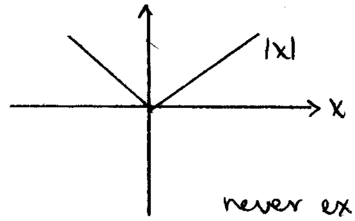
At $x=0$;

$$\lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

and

$$\lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

There is no limit as $h\to 0$ and so there is no tangent line at $x=0$.



Note that the problem is that $|x|$ has a sharp point at $x=0$. Tangent lines never exist at such points.

Also see the example at the bottom of the 2nd column on p. 102 of the book.

The Derivative

The slope of $f(x)$ at an arbitrary point x is:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Written this way this limit is itself a function called the derivative

Def The derivative of the function f with respect to the variable x is the function $f'(x)$ whose value is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Rates of Change

As discussed in section 1.1, the difference quotient

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

is the average rate of change of $f(x)$ in going from $x = x_0$ to $x = x_1$. The limit

$$\lim_{x \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

is the <u>instantaneous rate of change</u> of $f(x)$ at $x = x_0$. Note that if we set

$$h = x_1 - x_0$$

$$\Rightarrow x_1 = x_0 + h$$

then

$$\lim_{x \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$$

> So the instantaneous rate of change of $f(x)$ with respect to x at the point $x = x_0$ is just the derivative $f'(x)$ evaluated at $x = x_0$.

Example

A mold has a mass of $3t^2$ grams after t hours of growth. What is the instantaneous rate of growth when $t = 3$ hours?

Solution:

Let

$$M(t) = 3t^2$$

be the mass as a function of time. The instantaneous rate of change of the mass as a function of time is

$$M'(t) = \lim_{h \to 0} \frac{M(t+h) - M(t)}{h}$$

$$= \lim_{h \to 0} \frac{3(t+h)^2 - 3t^2}{h}$$

$$= \lim_{h \to 0} \frac{3t^2 + 6th + 3h^2 - 3t^2}{h}$$

$$= \lim_{h \to 0} (6t + 3h)$$

$$= 6t$$

At $t = 3$,

$$M'(3) = 6(3)\ \frac{\text{grams}}{\text{hour}}$$

$$= 18\ \frac{\text{gm}}{\text{hr}}$$

Note the units. If $M(t)$ is in grams, then

$$[M'(t)] = \left[\frac{\Delta M}{\Delta t}\right] = \frac{\text{gm}}{\text{hr}}$$

means "units of"

Example
(# 30, p.102)

What is the rate of change of the volume of a sphere of radius r with respect to the radius when the radius is $r=2$?

Solution:

$$V(r) = \frac{4}{3}\pi r^3$$

$$\Rightarrow V'(r) = \lim_{h \to 0} \frac{\frac{4}{3}\pi(r+h)^3 - \frac{4}{3}\pi r^3}{h}$$

$$= \frac{4}{3}\pi \lim_{h\to 0} \frac{\cancel{r^3} + 3r^2h + 3rh^2 + h^3 \cancel{-r^3}}{h}$$

$$= \frac{4}{3}\pi \lim_{h\to 0} (3r^2 + 3rh + h^2)$$

$$= \frac{4}{3}(3r^2)$$

$$= 4r^2$$

So

$$V'(2) = 4(2)^2 = 16$$