APPM 1350: Section 2.1: The Derivative of a Function

As we discussed in the last lecture, the derivative of $f(x)$ with respect to x at a point x_0 is the slope of the line tangent to $f(x)$ at x_0. More generally, the derivative is a function $f'(x)$ that gives the slope of $f(x)$ at any point x (provided $f'(x)$ exists at x). Formally;

> Def: The derivative of the function f with respect to the variable x is the function $f'(x)$ whose value at x is
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
>
> provided the limit exists.

If $f'(x)$ exists, we say that $f(x)$ is differentiable at x (or that $f(x)$ has a derivative at x).

Example

Find the derivative of $f(x) = (x-4)^2$

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h-4)^2 - (x-4)^2}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 8(x+h) + \cancel{16} - x^2 + 8x - \cancel{16}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{8x} - 8h - \cancel{x^2} + \cancel{8x}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 8h}{h}$$

$$= \lim_{h \to 0} (2x - 8 + h)$$

$$= 2x - 8$$

$$= 2(x-4)$$

Example

Find the derivative of $f(x) = \sqrt{x}$. What is the domain of $f(x)$ and $f'(x)$? Are they the same?

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{x+h - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Domain of $f(x) = \sqrt{x}$ is $x \geq 0$
Domain of $f'(x) = x > 0$.
They are <u>not</u> the same!

The previous example illustrates a crucial point.

> The domain of $f'(x)$ may not necessarily be the same as the domain of $f(x)$.

Notations for the Derivative

All of the following notations mean the derivate of $y = f(x)$:

$y'(x)$	"y prime"
$\frac{dy}{dx}$	"dy dx"
$\frac{df}{dx}$	"df dx"
$\frac{d}{dx} f(x)$	"d-dx of f(x)"
$D_x f(x)$	"Dx of f"
$\dot{y}(x)$	"y dot"

Example

Find $\frac{d}{dx}f(x)$ given $f(x) = \frac{x+3}{x-2}$. What is the domain of $f(x)$ and $f'(x)$?

Solution:

$$\frac{d}{dx}\left(\frac{x+3}{x-2}\right) = \lim_{h\to 0} \frac{\left(\frac{x+h+3}{x+h-2}\right) - \left(\frac{x+3}{x-2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h+3)(x-2)}{(x+h-2)(x-2)} - \frac{(x+3)(x+h-2)}{(x-2)(x+h-2)}\right]$$ bring everything over a common denominator

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h+3)(x-2) - (x+3)(x+h-2)}{(x+h-2)(x-2)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(\cancel{x^2} - \cancel{2x} + \cancel{hx} - 2h + \cancel{3x} - \cancel{6}) - (\cancel{x^2} + \cancel{hx} - \cancel{2x} + \cancel{3x} + 3h - \cancel{6})}{(x+h-2)(x-2)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-5h}{(x+h-2)(x-2)}\right]$$

$$= \lim_{h\to 0} \frac{-5}{(x+h-2)(x-2)}$$

$$= -\frac{5}{(x-2)^2}$$

The domain of $f(x)$ is $\{x \in \mathbb{R} : x \neq 2\}$. The domain of $f'(x)$ is the same.

Example

Find the equation for the line tangent to $f(x)$ in the previous example at an orbitrary point $x_0 \neq 2$.

Solution:

From the previous page:

$$f(x_0) = \frac{x_0 + 3}{x_0 - 2}$$

and

$$f'(x_0) = m = -\frac{5}{(x_0-2)^2}$$

So

$$y_t(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= \frac{x_0+3}{x_0-2} - \frac{5}{(x_0-2)^2}(x - x_0)$$

Differentiable on an Interval

Def A function $y = f(x)$ is differentiable on an open interval (finite or infinite) is $f'(x)$ exists at each point in the interval. It is differentiable on the closed interval $[a,b]$ if it is differentiable on (a,b) and the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \text{(Right-hand derivative at a)}$$

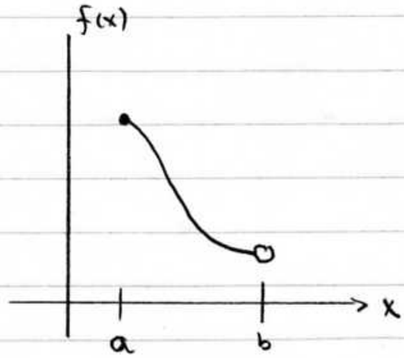
and

$$\lim_{x \to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \text{(Left-hand derivative at b)}$$

exist.

Example

Is $f(x)$ given by



differentable on $[a,b]$?

Solution:

No. There is no left-handed derivative at $x=b$.

An Alternative Form for Calculating Derivatives

We have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

But let $x = c+h$. Then $x - c = h$ and

$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}}$$

Example

The volume of a right-circular cylinder of radius R and height H is:

$$V = \pi R^2 H$$

Let $R = 2$ cm. How fast does the volume change as a function of height at the point where $H = 3$ cm?

Solution:

The rate of change is given by the derivative. Since $R = 2$cm, we have:

$$V(H)\Big|_{R=2} = \pi (2)^2 H \qquad \text{cm}^3$$

$$= 4\pi H \qquad \text{cm}^3$$

Then at $H = 3$ cm:

$$V'(3) = \lim_{H \to 3} \frac{V(H) - V(3)}{H - 3}$$

$$= \lim_{H \to 3} \frac{4\pi H - 4\pi(3)}{H - 3}$$

$$= \lim_{H \to 3} \frac{4\pi(H-3)}{(H-3)}$$

$$= 4\pi \qquad \frac{\text{cm}^3}{\text{cm}}$$

Example

Same problem as in the previous example, but how fast is the volume changing as a function of radius around R = 2cm and H = 3cm?

Solution:

$$V(R)\Big|_{H=3} = \pi R^2 (3)$$

$$= 3\pi R^2 \qquad cm^3$$

Then at R = 2:

$$V'(2) = \lim_{R \to 2} \frac{V(R) - V(2)}{R-2}$$

$$= \lim_{R \to 2} \frac{3\pi R^2 - 3\pi(2)^2}{R-2}$$

$$= \lim_{R \to 2} 3\pi \left(\frac{R^2-4}{R-2}\right)$$

$$= 3\pi \lim_{R \to 2} \frac{(R-2)(R+2)}{(R-2)}$$

$$= 3\pi \lim_{R \to 2} (R+2)$$

$$= 3\pi (2+2)$$

$$= 12\pi \quad \frac{cm^3}{cm}$$

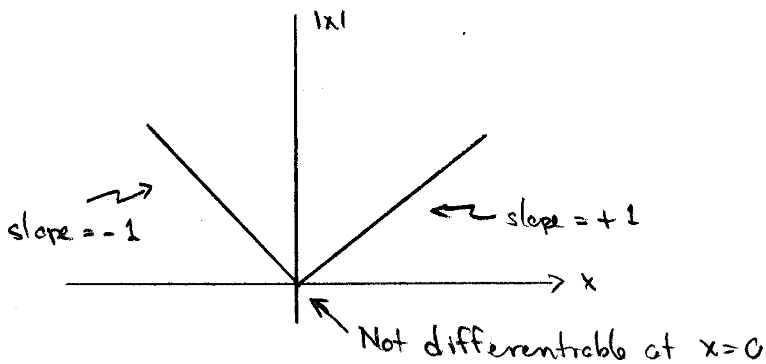
A Couple Important Theorems

Thm: If $f(x)$ has a derivative at $x=c$, then $f(x)$ is continuous at $x=c$.

But note!! Just because a function is continuous at $x=c$ does not mean $f'(c)$ exists.

Example

$f(x)=|x|$ is continuous at $x=0$ but it is not differentiable because its slope on either side of $x=0$ is different (hence the limit of the difference quotient doesn't exist).

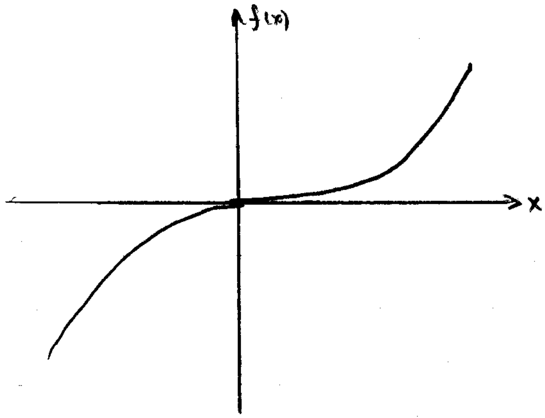


Thm Intermediate Value Theorem for Derivatives. If a and b are any two points in an interval on which f is differentiable, then f takes on every value between $f'(a)$ and $f'(b)$.

Matching Graphs of Functions to Graphs of Derivatives

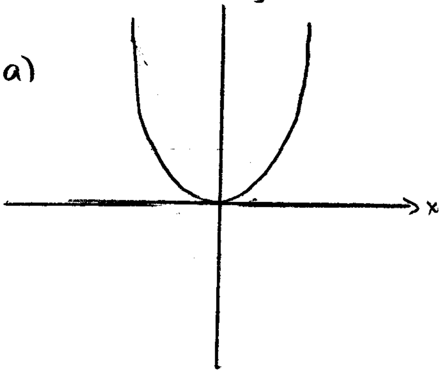
Example

Consider the function $f(x) = x^3$

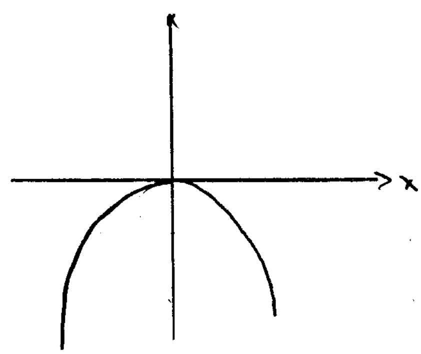


Which of the following is a graph of $f'(x)$?

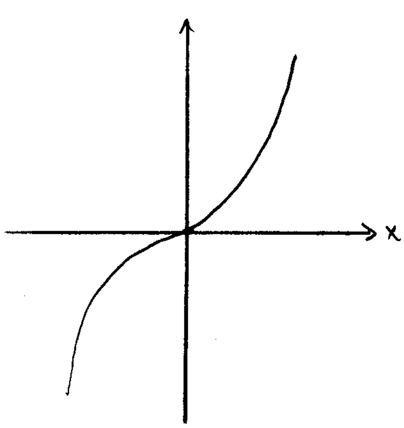
a)



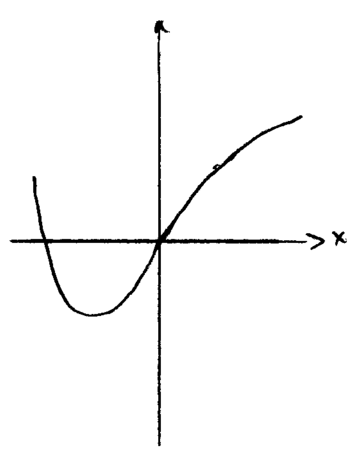
b)



c)



d)



Solution:

Look at the sign of the slope of f(x) in different regions:

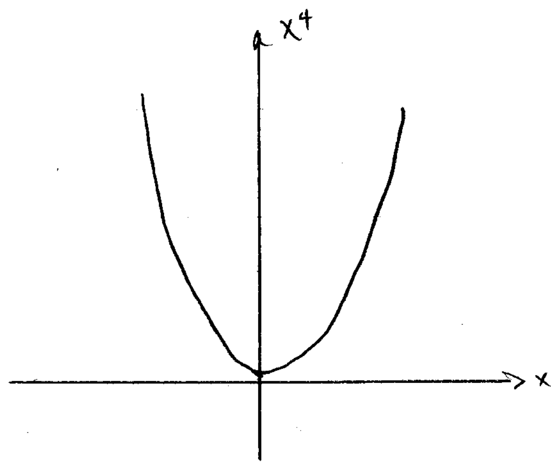
$$\frac{d}{dx}x^3 = \begin{cases} >0 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ >0 & \text{if } x>0 \end{cases}$$

The only graph that is either positive or zero is a). Also, note that f(x) has an anti-symmetry wrt $x=0$. We would expect f'(x) to have either a symmetry or anti-symmetry wrt $x=0$. Graph a) exhibits just that.

<u>Example</u>

Consider $f(x) = x^4$. Which of the graphs on the previous page is a graph of $f'(x)$?

Solution:



From the graph,

$$\frac{d}{dx}x^4 = \begin{cases} <0 & x<0 \\ 0 & x=0 \\ >0 & x>0 \end{cases}$$

This eliminates graphs a) and b). Graph d) won't work also because it is positive for some negative x. So the answer is c) by elimination. Also note the symmetry/anti-symmetry of f(x) and f'(x) around $x=0$.