APPM 1350: Section 2.4: Derivatives of Trig Functions

Some Interesting and Useful Inequalities and Limits

Theorem

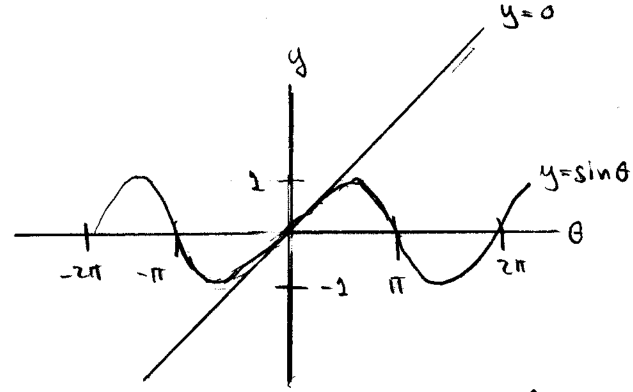
$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

I have given two proofs of this in previous notes. I also highly recommend the proof given in the book on pages 144-145, it is actually less complicated than my proofs.

Another very useful inequality is:

Theorem

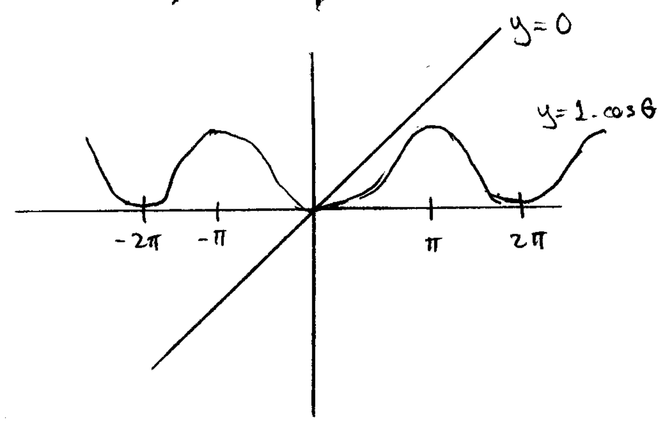
$$|\sin\theta| \leq |\theta|$$



One other useful inequality:

Theorem:

$$|1-\cos\theta| \leq |\theta|$$



With these theorems we can prove another useful theorem:

<u>Theorem</u>

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

Proof:

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} - \frac{2\sin^2\left(\frac{h}{2}\right)}{h} \qquad \text{since } \cos h = 1 - 2\sin^2\left(\frac{h}{2}\right)$$

$$= \lim_{\alpha \to 0} - \frac{\sin^2 \alpha}{\alpha} \qquad \text{letting } \alpha = \frac{h}{2}$$

$$= -\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} \lim_{\alpha \to 0} \sin \alpha$$

$$= -1 \cdot 0$$

$$= 0$$

The Derivative of the Sine & Cosine

With these theorems we are in a position to derive the derivative of the sine function.

$$\boxed{\frac{d}{dx} \sin x = \cos x}$$

Proof:

$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{(\sin x\cos h+\cos x\sin h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h-1)+\cos x\sin h}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h-1)}{h} + \lim_{h\to 0} \frac{\cos x\sin h}{h}$$

$$= \sin x \lim_{h\to 0} \frac{\cos h-1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin x \cdot 0 + \cos x \cdot 1$$

$$= \cos x$$

In a very similiar way, we can show that

$$\boxed{\frac{d}{dx}\cos x = -\sin x}$$

Example

Find $\frac{d}{dx}(3x^2 + 2\sin x)$

Solutron,

$$\frac{d}{dx}(3x^2 + 2\sin x) = \frac{d}{dx}3x^2 + \frac{d}{dx}2\sin x$$

$$= 3\frac{d}{dx}x^2 + 2\frac{d}{dx}\sin x$$

$$= 3(2x) + 2\cos x$$

$$= 6x + 2\cos x$$

Example

Find $\frac{d}{dx}\sin^2 x$

Solution:

$$\frac{d}{dx}\sin x = \frac{d}{dx}\underbrace{(\sin x)}_{u}\underbrace{(\sin x)}_{v}$$

$$\frac{d}{dx}uv = u\frac{dv}{dt} + v\frac{du}{dx}$$

$$= \sin x(\cos x) + \sin x(\cos x)$$

$$= 2\sin x\cos x$$

The Derivative of the Tangent

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

This is easy to prove given the derivative of the sine and cosine and the quotient rule.

Proof:

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

$$= \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

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The Remaining Trig Derivatives

The derivatives of the reciprocal trig functions follow from the derivatives of the sine, cosine, and tangent and the quotient rule.

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

Class exercise

Prove $\quad \frac{d}{dx}\cot x = -\csc^2 x$

Proof:

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{1}{\tan x}$$

$$= \frac{\tan x \frac{d}{dx}1 - 1\frac{d}{dx}\tan x}{\tan^2 x}$$

$$= -\frac{\sec^2 x}{\tan^2 x}$$

$$= -\frac{1}{\cos^2 x}\frac{\cos^2 x}{\sin^2 x}$$

$$= -\csc^2 x$$

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Example

(#20, p152)

Find $\frac{d}{d\theta}(1+\sec\theta)\sin\theta$

Solution:

$$\frac{d}{d\theta}(1+\sec\theta)\sin\theta = \sin\theta\frac{d}{d\theta}(1+\sec\theta) + (1+\sec\theta)\frac{d}{d\theta}\sin\theta$$

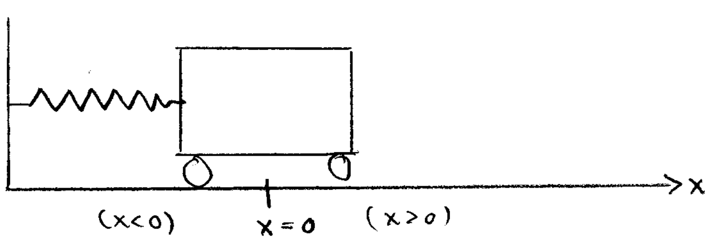
$$= \sin\theta\left[\cancelto{0}{\frac{d}{d\theta}1} + \frac{d}{d\theta}\sec\theta\right] + (1+\sec\theta)\frac{d}{d\theta}\sin\theta$$

$$= \sin\theta(\sec\theta\tan\theta) + (1+\sec\theta)(\cos\theta)$$

$$= \tan^2\theta + \cos\theta + 1$$

$$= \sec^2\theta + \cos\theta \qquad \text{since } 1+\tan^2\theta = \sec^2\theta$$

Simple Harmonic Motion



Consider a wheeled cart that is connected to a wall with a spring. When the cart is at rest the spring is unstretched. Call this position $x=0$. If we move the cart toward the wall $(x<0)$, the spring will compress and try to push the cart back toward the right. Alternately, if we move the cart away from the wall $(x>0)$

then the spring will be stretched and will try to pull the cart back to the left. The motion of the cart will be of the form

$$x(t) = a\cos t + b\sin t \qquad (1)$$

where a and b are constants (strictly speaking this assumes a spring constant equal to 1 and that the cart have mass 1. We also assume no friction or air resistence). The solution $x(t)$ given by (1) will be <u>oscillatory</u> and is called <u>simple harmonic motion</u>.

<u>Example</u>

Assume $x(0) = x_0$ and $\dot{x}(0) = 0$, find a and b in (1).

Solution:

This says that at time zero the cart has been pulled to x_0 and then released from rest $(\dot{x}(0) = v(0) = 0)$. So from (1):

$$x_0 = x(0) = a\cos 0 + b\sin 0$$
$$= a$$

$$\Rightarrow a = x_0$$

and differentiating (1):

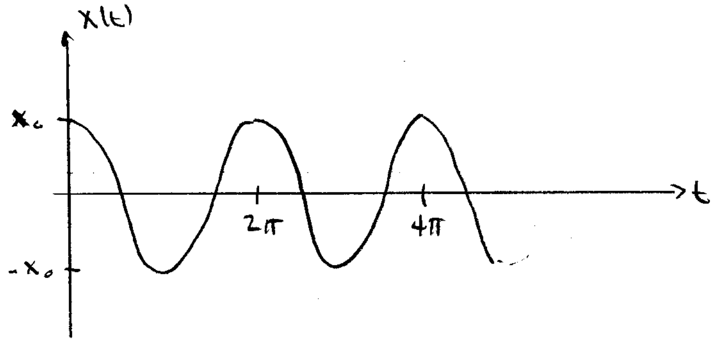
$$\dot{x}(t) = -a\sin t + b\cos t$$

$$\Rightarrow 0 = \dot{x}(0) = -a\sin 0 + b\cos 0$$
$$= b$$

$$\Rightarrow b = 0$$

Hence:

$$x(t) = x_0 \cos t$$



The position of the cart will oscillate between the points $\pm x_o$. The period of the oscillation will be 2π seconds, this is the time required for the cart to go from $x = x_o$ back to $x = x_o$.

Many physical and biological systems exhibit simple harmonic behavior. These include pendulums (assuming small oscillations), certain electrical circuits (voltage oscillates harmonically), and phonons in an electromagnetic field.

More Limits

We started this section with some theorems about trig inequalities and limits. We used these to derive the derivative of the sine (and cosine), and then used these and the quotient rule to derive the derivatives of the other trig functions (or at least alluded to how to derive them). We now pick back up with limits and exploring ways of solving limit problems involving trig functions We do this by example.

Example
(# 35, p. 152)

Find $\lim_{y \to 0} \frac{\sin 3y}{4y}$

Solution: $\lim_{y \to 0} \frac{\sin 3y}{4y} = \frac{1}{4} \lim_{y \to 0} \frac{\sin 3y}{y}$

$$= \frac{1}{4} \lim_{y \to 0} \frac{3 \sin 3y}{3y}$$

$$= \frac{3}{4} \lim_{y \to 0} \frac{\sin 3y}{3y}$$

$$= \frac{3}{4} \lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha}$$

Let $\alpha = 3y$ and note that $y \to 0$ means $\alpha \to 0$

$$= \frac{3}{4} \cdot 1$$

$$= \frac{3}{4}$$

Example
(# 39, p. 152)

Find $\lim_{x \to 0} \frac{x \csc 2x}{\cos 5x}$

Solution: $\lim_{x \to 0} \frac{x \csc 2x}{\cos 5x} = \lim_{x \to 0} \frac{x}{(\cos 5x)(\sin 2x)}$

$$= \cancelto{1}{\lim_{x \to 0} \frac{1}{\cos 5x}} \lim_{x \to 0} \frac{x}{\sin 2x}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{2x}{\sin 2x}$$

$$= \frac{1}{2}$$

(use the trick above, let $\alpha = 2x$)

Tangent Lines

Finally, we compute the equation of tangent lines, but this time on equations involving trig functions.

Example
(#56, p 153)

Does $y = x + 2\cos x$ have any horizontal tangents in $x \in [0, 2\pi]$? If so, where?

Solution:

A horizontal tangent is where the slope is zero, hence $y'(x) = 0$. So:

$$y = x + 2\cos x$$

$$y'(x) = 1 - 2\sin x$$

$$\Rightarrow \quad 0 = 1 - 2\sin x$$

$$\Rightarrow \sin x = \frac{1}{2}$$

$$\Rightarrow x = \frac{\pi}{6} \quad \text{and} \quad x = \frac{5\pi}{6}$$

Example (#57, p. 153)

Find all $x \in (-\pi/2, \pi/2)$ where $y = \tan x$ has a tangent line that is parallel to $y = 2x$.

Solution:

$y = \tan x$ will have a tangent line parallel to $y = 2x$ when the slope of $y = \tan x$ is equal to 2 (since this is the slope of $y = 2x$). Thus:

$$y(x) = \tan x$$

$$\frac{dy}{dx} = \sec^2 x$$

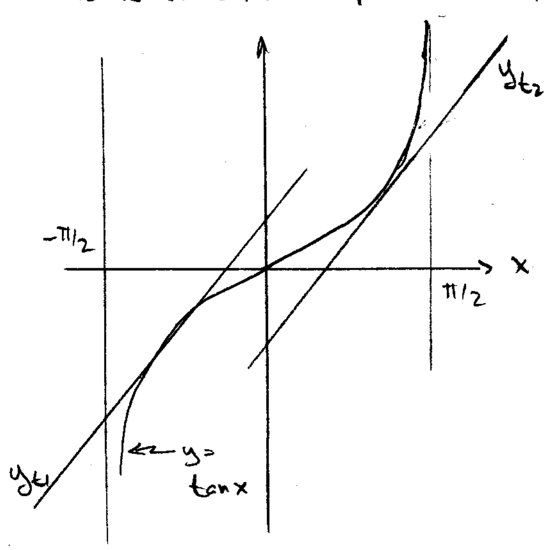
$$\Rightarrow \quad 2 = \sec^2 x$$

$$\Rightarrow \quad \cos^2 x = \frac{1}{2}$$

$$\Rightarrow \quad \cos x = \pm\sqrt{\frac{1}{2}}$$

$$= \pm \frac{\sqrt{2}}{2}$$

For $x \in (-\pi/2, \pi/2)$ this is true for $\pm \frac{\pi}{4}$. The equation of the tangent lines at these points is:



$$y_{t_1}(x) = -1 + 2\left(x + \frac{\pi}{4}\right) \quad \text{At } x = -\frac{\pi}{4}$$

$$y_{t_2}(x) = 1 + 2(x - \pi/4) \quad \text{At } x = \frac{\pi}{4}$$