APPM 1350: Sections 3.1 & 3.2: Extreme Values and the Mean Value Theorem

Minimum and Maximum Points

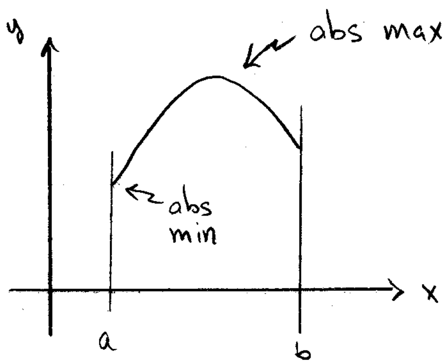
Def Let f be a function with domain D. Then f has an absolute maximum value on D at the point c if

$$f(x) \le f(c) \qquad \forall\, x \in D$$

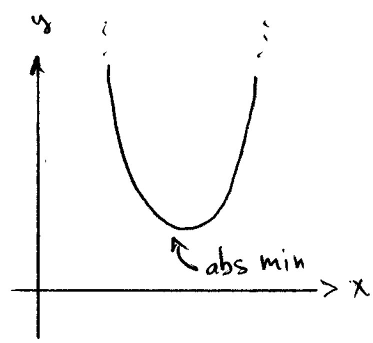
and an absolute minimum value on D at c if

$$f(x) \ge f(c) \qquad \forall\, x \in D$$

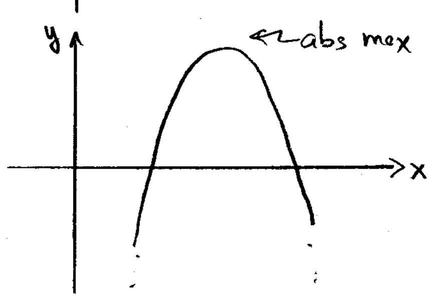
Examples



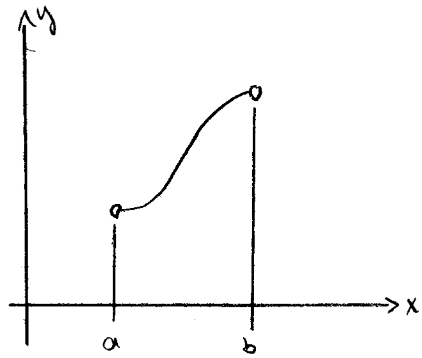
$D = \{x \in \mathbb{R} : a \le x \le b\}$



$D = \mathbb{R}$

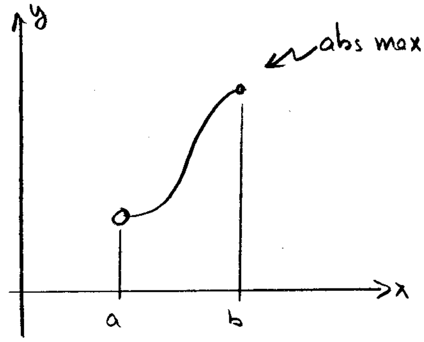


$D = \mathbb{R}$



$D = \{x \in \mathbb{R} : a < x < b\}$

No abs min or max



$D = \{x \in \mathbb{R} : a < x \leq b\}$

Only an abs max

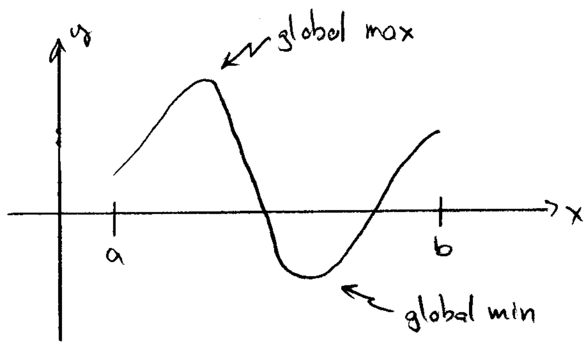
Def A function f has a <u>local maximum</u> value at an <u>interior point</u> c of its domain if

$$f(x) \leq f(c) \quad \forall\ x \text{ in some } \underline{\text{open}} \text{ interval containing } c$$

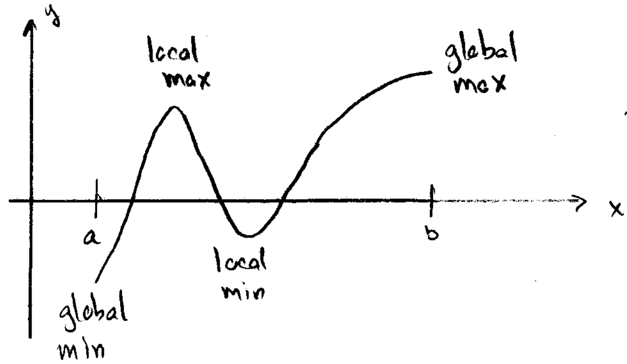
A function f has a <u>local minimum</u> value at an <u>interior point</u> c of its domain if

$$f(x) \geq f(c) \quad \forall\ x \text{ in some } \underline{\text{open}} \text{ interval containing } c$$

<u>Examples</u>



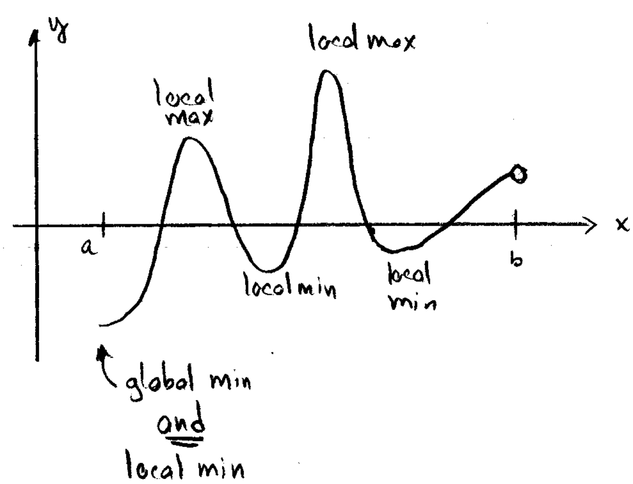
$D = \{x \in \mathbb{R} : a \leq x \leq b\}$



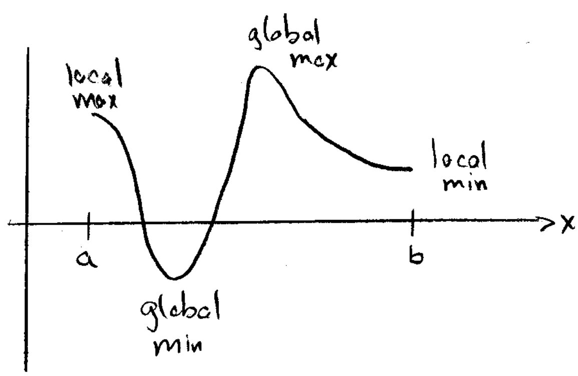
$D = \{x \in \mathbb{R} : a \leq x \leq b\}$

Def f has a local minimum or local maximum at an endpoint c if the appropriate inequality holds for all x in some half-open interval in its domain containing c.

Examples



$D = \{x \in \mathbb{R} : a \leq x < b\}$



$D = \{x \in \mathbb{R} : a \leq x \leq b\}$

Def Local and global minimum and maximum are called <u>extreme values</u> or <u>extrema.</u>

Finding Extrema

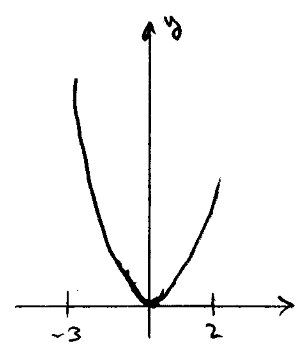
<u>The First Derivative Theorem for Local Extrema</u> If f has a local minimum or maximum value at an interior point c of its domain and if f' is defined at c, then

$$f'(c) = 0$$

Extrema for differential functions are where the function "turns around". The slope will be zero at that point $\Rightarrow f'(c) = 0$. However, note that you can have extrema at points where f' might not exist (for example $|x|$ has a global min at $x=0$ but is not differentiable at that point). So $f'(c)=0$ is a <u>sufficient but not necessary</u> condition for extrema.

Example

Find the extreme values of $f(x) = x^4$ on $[-3, 2]$



Soln: $$f'(x) = \frac{d}{dx} x^4 = 4x^3$$

$$f'(x) = 0 \Rightarrow 4x^3 = 0 \Rightarrow x = 0$$

This is a local minimum. $f(-3) = 81$ is both a local and global max. $f(2) = 16$ is a local max.

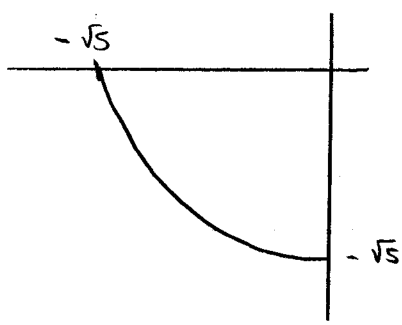
Def An interior point of the domain of a function f where f' is zero or undefined is a critical point of f

Observation: The only domain points where a function can assume extreme values are critical points and endpoints.

More Examples

Example

$$g(x) = -\sqrt{5 - x^2} \qquad x \in [-\sqrt{5}, 0]$$



Local/global max at $x = -\sqrt{5}$
Local/global min at $x = 0$

$$g'(x) = \frac{d}{dx} - (5 - x^2)^{1/2}$$

$$= -\frac{1}{2}(5 - x^2)^{-1/2}(-2x)$$

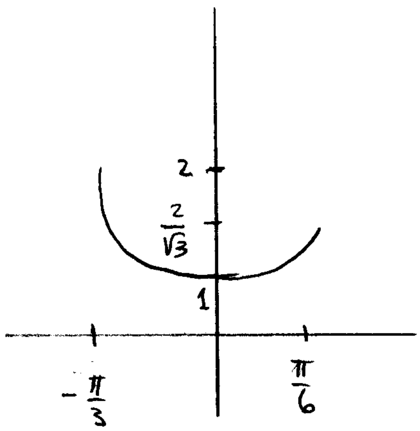
$$= \frac{x}{\sqrt{5 - x^2}}$$

This exists $\forall\ x \in (-\sqrt{5}, 0)$.

$g'(x) \neq 0 \quad \forall\ x \in (-\sqrt{5}, 0)$, so no critical points in the interior

Example

$g(x) = \sec x \qquad x \in \left[-\frac{\pi}{3}, \frac{\pi}{6}\right]$



$g\left(-\frac{\pi}{3}\right) = \sec\left(-\frac{\pi}{3}\right) = 2$

$g\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$

$$g'(x) = \frac{d}{dx}\sec x$$

$$= \sec x \tan x$$

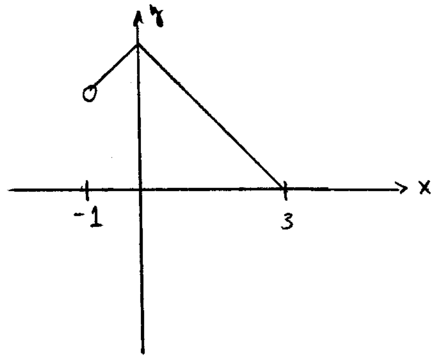
This exists $\forall\ x \in \left[-\frac{\pi}{3}, \frac{\pi}{6}\right]$. Now see if there are any critical points.

$$g'(x) = 0 \Rightarrow \sec x \tan x = 0 \Rightarrow x = 0$$

So $x=0$ is a critical point. $g(0) = \sec 0 = 1$. So the point $(0,1)$ is a global <u>and</u> local minimum. The point $\left(-\frac{\pi}{3}, 2\right)$ is a global <u>and</u> local maximum. The point $\left(\frac{\pi}{6}, \frac{2}{\sqrt{3}}\right)$ is a local maximum.

Example

$$f(x) = 3 - |x| \qquad x \in (-1, 3]$$

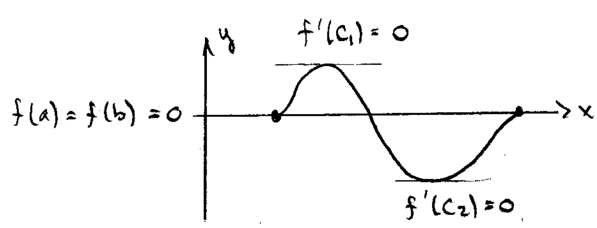


$f(3) = 0$ is both a local and global minimum. The function is not defined at the left hand endpoint, so that point is not a local minimum. The point $x=0$ is a local and global maximum, but it is not differentiable there (hence $f'(0)$ is undefined and $x=0$ is a critical point).

Rolle's Theorem

> **Rolle's Thm** Suppose that $y=f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable at every point of its interior (a,b). If
> $$f(a) = f(b) = 0$$
> then there is at least one $c \in (a,b)$ at which
> $$f'(c) = 0$$

Example



1. $f(a) = f(b)$
2. $f(x)$ diff on (a,b) and cont. on $[a,b]$
3. In this case there are two points in (a,b) where $f'(c)=0$. Rolle's thm guarantees at least one.

The Mean Value Theorem

The Mean Value Theorem Suppose $y=f(x)$ is continuous on a closed interval $[a,b]$ and differentiable on the interval's interior (a,b). Then there is at least one $c \in (a,b)$ at which

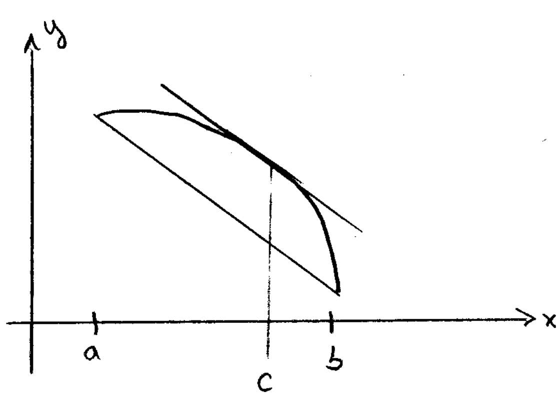
$$\frac{f(b)-f(c)}{b-a} = f'(c)$$

The difference quotient

$$\frac{f(b)-f(a)}{b-a}$$

is the slope of the line from the point $(a, f(a))$ to $(b, f(b))$. What the mean value theorem says is that if f is continuous on $[a,b]$ and differentiable on (a,b), then somewhere in (a,b) the tangent line to $f(x)$ will be parallel to the line between the endpoints. Put differently, somewhere in (a,b) the instantaneous rate of change of $f(x)$ will equal the average rate of change over the entire interval.

Example



Example

$$f(x) = x^{2/3} \qquad x \in [-1, 8]$$

$$f(-1) = 1$$

$$f(8) = 4$$

$$\Rightarrow \frac{f(8) - f(-1)}{8 - (-1)} = \frac{4 - 1}{9} = \frac{1}{3}$$

Now; $f'(x) = \frac{2}{3} x^{1/3}$. So where is $f'(x) = \frac{1}{3}$?

$$\frac{2}{3} x^{1/3} = \frac{1}{3}$$

$$\Rightarrow x^{1/3} = \frac{1}{2}$$

$$\Rightarrow x = \left(\frac{1}{2}\right)^3$$

$$= \frac{1}{8}$$

So $f'\left(\frac{1}{8}\right) = \frac{f(8) - f(1)}{8 - (-1)}$ and $\frac{1}{8} \in (-1, 8)$

Thm If $f'(x) = 0$ for $\forall\, x \in I$, then $f(x) = C$ for $\forall\, x \in I$ where C is a constent.

Thm If $f'(x) = g'(x)$ $\forall\, x \in I$ then $\exists\, C \ni f(x) = g(x) + C$ $\forall\, x \in I$ where C is a constent.

Glossary of the notation above: $\forall$ means "for all" $\in$ means "is an element in" $\exists$ means "there exists" $\ni$ means "such that"

Consider $f(x) = \tan x$

$g(x) = \tan x + 5$

clearly

$$f'(x) = \sec^2 x$$
$$g'(x) = \sec^2 x$$

So $f(x) = g(x) + C$ and $f'(x) = g'(x)$. What our second theorem says, however, is the opposite. <u>Anytime</u> $f'(x) = g'(x)$, then

$$g(x) = f(x) + c$$

So if $f'(x) = g'(x) = \sec^2 x$, we can guorontee that

$$g(x) = \underbrace{\tan x}_{f(x)} + c$$

c = some constent.

<u>Example</u>

Suppose

$$f'(x) = x^3$$

What is a $f(x)$ such that $f'(x) = x^3$? We know that

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\Rightarrow \frac{1}{n} \frac{d}{dx} x^n = x^{n-1}$$

$$\Rightarrow \frac{d}{dx} \underbrace{\left(\frac{1}{n} x^n\right)}_{f(x)} = \underbrace{x^{n-1}}_{f'(x)}$$

So if $f'(x) = x^3$ then $n-1 = 3 \Rightarrow n = 4$ and $f(x) = \frac{1}{4}x^4$. Now, suppose we have

$$f'(x) = g'(x) = x^3$$

Then from the second theorem on the previous page

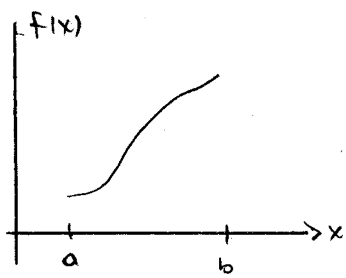
$$g(x) = f(x) + c$$

$$= \frac{1}{4}x^4 + c$$

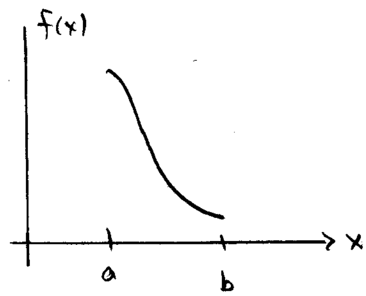
where C is some (unspecified) constent.

Increasing and Decreasing Functions

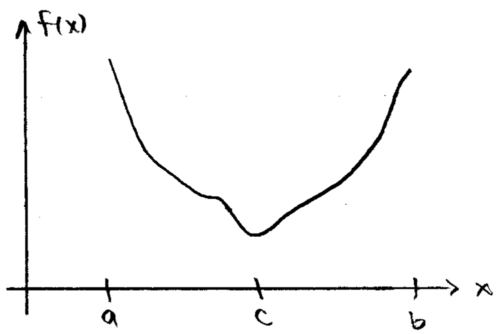
Def Let f be a function defined on an interval I. f is said to be increasing on I if $x_1 < x_2$ implies $f(x_1) < f(x_2)$ for all pairs of points x_1 and x_2 in I. If $x_1 < x_2$ implies $f(x_1) > f(x_2)$ for all pairs of points x_1 and x_2 in I then f is said to be decreasing on I.



$f(x)$ is increasing on $[a,b]$



$f(x)$ is decreasing on $[a,b]$



$f(x)$ is decreasing on $[a,c]$

$f(x)$ is increasing on $[c,b]$

The definition doesnt apply on the whole interval $[a,b]$, only on the subintervals $[a,c]$ and $[c,b]$.

<u>The First Derivative Test for Increasing and Decreasing</u> Suppose that f is continuous on $[a,b]$ and differentiable on (a,b).

If $f' > 0$ at <u>every</u> point in (a,b) then f <u>increases</u> on $[a,b]$

If $f' < 0$ at <u>every</u> point in (a,b) then f <u>decreases</u> on $[a,b]$

<u>Example</u>

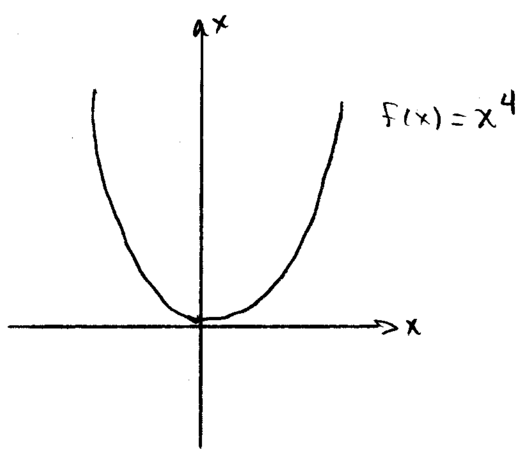
Consider

$$f(x) = x^4$$

on $I = [-4, 5]$. Clearly $f(x)$ is differentiable on $(-4,5)$ and continuous on $[-4,5]$. Further

$$f'(x) = 4x^3$$

$f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$. So $f(x) = x^4$ is decreasing on $[-4, 0]$ and increasing on $[0, 5]$.

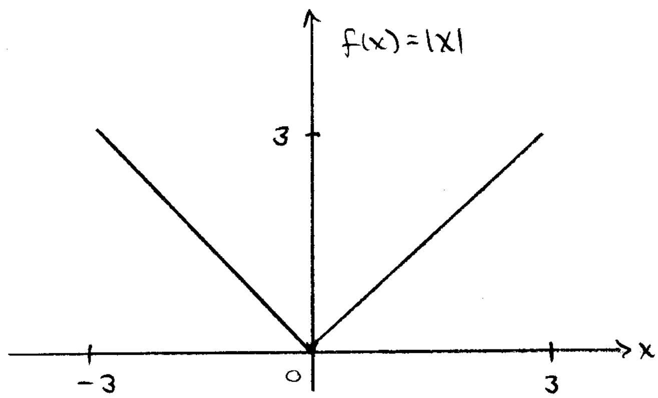


Example

Consider

$$f(x) = |x|$$

on $I = [-3, 3]$. $f(x)$ is not differentiable at $x=0$, but is differentiable on $(-3,0)$ and $(0,3)$. Further, it is continuous on $[-3,0]$ and $[0,3]$. For $x \in (-3,0)$ the derivative $f'(x) = -1$ and for $x \in (0,3)$ the derivative $f'(x) = 1$. So by our theorem $f(x) = |x|$ is decreasing on $[-3,0]$ and increasing on $[0,3]$.



Miscelleneous Examples

Example (# 18, p. 203)

Show that $|\sin b - \sin a| \leq |b-a|$ for any numbers a and b

Soln: The function $f(x)=\sin x$ is continuous and differentiable for all $x \in \mathbb{R}$, hence for any $[a,b]$ where $a<b$. Then by the mean value theorem, there is some $c \in (a,b)$ where

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \frac{\sin b - \sin a}{b-a} = \cos(c)$$

$$\Rightarrow \left|\frac{\sin b - \sin a}{b-a}\right| = |\cos c| \leq 1$$

$$\Rightarrow |\sin b - \sin a| \leq |b-a|$$

Example (# 38b, p. 204)

Find all possible functions with the derivative $y' = \frac{1}{\sqrt{x}}$

Soln:

$$f'(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$$

From the example on p. 11 of these notes we have $n-1 = -\frac{1}{2} \Rightarrow n = \frac{1}{2}$.
Hence

$$f(x) = 2x^{1/2} = 2\sqrt{x}$$

However, any function $g(x) = 2\sqrt{x} + c$ (c = const) will have the derivative $1/\sqrt{x}$, so it is $g(x)$ that represents all functions of interest.