APPM 1350: Section 3.5: Limits as $x \to \pm\infty$, Asymptotes and Dominant Terms

We have discussed

$$\lim_{x \to a} f(x) = L$$

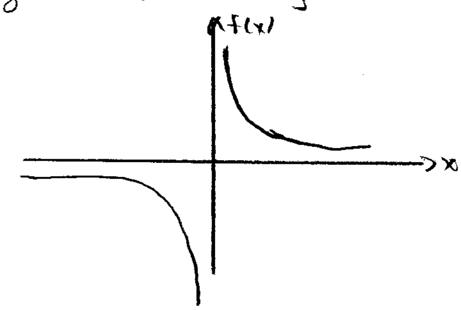
where a is finite and L is either finite or infinite. So for example;

$$\lim_{x \to 2} \frac{x-2}{x+2} = 0$$

or

$$\lim_{x \to 2} \frac{x+2}{|x-2|} = +\infty$$

We now extend these ideas to cases where $x \to \pm\infty$. This simply is the limit, if it exists, as x becomes larger and larger in either a negative or positive direction. So, for example,



$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

$$\lim_{x \to \infty} (3x^4 - x^2 + x - 7) = \lim_{x \to \infty} x^4\left(3 - \frac{1}{x^2} + \frac{1}{x^3} - \frac{7}{x^4}\right)$$

$$= \underbrace{\left(\lim_{x \to \infty} x^4\right)}_{= \infty} \underbrace{\left(\lim_{x \to \infty} 3 - \frac{1}{x^2} + \frac{1}{x^3} - \frac{7}{x^4}\right)}_{3}$$

$$= \infty$$

$$\lim_{x \to +\infty} \frac{2x+5}{x^2 - 7x + 3} = \lim_{x \to \infty} \frac{x\left(2 + \frac{5}{x}\right)}{x^2\left(1 - \frac{7}{x} + \frac{3}{x^2}\right)}$$

$$= \left(\lim_{x \to \infty} \frac{1}{x}\right)\left(\lim_{x \to \infty} \frac{2 + 5/x}{1 - 7/x + 3/x^2}\right)$$

$$= 0 \cdot 2 = 0$$

Def We say that $f(x)$ has a limit L as $x \to +\infty$ and write

$$\lim_{x \to \infty} f(x) = L$$

if $\forall \epsilon > 0 \ \exists M$ such that $\forall x$

$$x > M \implies |f(x) - L| < \epsilon$$

We say $f(x)$ has a limit L is $x \to -\infty$ and write

$$\lim_{x \to -\infty} f(x) = L$$

if $\forall \epsilon > 0 \ \exists N$ such that $\forall x$

$$x < N \implies |f(x) - L| < \epsilon$$

Limits at infinity behave pretty much like any other limit.

If $\lim\limits_{x \to \pm\infty} f(x) = L$ and $\lim\limits_{x \to \pm\infty} g(x) = L$

1. $\lim\limits_{x \to \pm\infty} [f(x) \pm g(x)] = L \pm M$

2. $\lim\limits_{x \to \pm\infty} f(x)\, g(x) = L \cdot M$

3. $\lim\limits_{x \to \pm\infty} k f(x) = kL$ $\qquad$ (k = const)

4. $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$

5. If $m, n \in \mathbb{Z}$, then $\lim\limits_{x \to \pm\infty} [f(x)]^{m/n} = L^{m/n}$ provided $L^{m/n} \in \mathbb{R}$.

Example

$$\lim_{x\to\infty} \frac{3x^3-4x+2}{7x^3+5} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(3-\frac{4}{x^2}+\frac{2}{x^3}\right)}{\cancel{x^3}\left(7+\frac{5}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{3-\frac{4}{x^2}+\frac{2}{x^3}}{7+\frac{5}{x^3}}$$

$$= \frac{3}{7}$$

Note that the numerator and denominator are polynomials of the same degree. The limits at $\pm\infty$ will always be the ratio of the leading coefficients.

Example

$$\lim_{x\to-\infty} \frac{4x^5+1}{3x^2+7} = \lim_{x\to-\infty} \frac{x^5\left(4+\frac{1}{x^5}\right)}{x^2\left(3+\frac{7}{x^2}\right)}$$

$$= \lim_{x\to-\infty} x^3 \lim_{x\to-\infty} \left(\frac{4+\frac{1}{x^5}}{3+\frac{7}{x^2}}\right)$$

$$= -\infty \cdot \frac{4}{3}$$

$$= -\infty$$

Note the numerator is a polynomial of degree larger than the degree of the polynomial in the denominator. These cases always have a limit of $\pm\infty$ as x goes to $-\infty$ or $+\infty$.

Example

$$\lim_{x\to-\infty} \frac{3x^2+4x}{5x^3+2x^2+x-20} = \lim_{x\to-\infty} \frac{x^2\left(3+\frac{4}{x}\right)}{x^3\left(5+\frac{2}{x}+\frac{1}{x^2}-\frac{20}{x^3}\right)}$$

$$= \left(\lim_{x\to-\infty} \frac{1}{x}\right)\left(\lim_{x\to-\infty} \frac{3+4/x}{5+2/x+1/x^2-20/x^3}\right)$$

$$= 0 \cdot \frac{3}{5}$$

$$= 0$$

Numerator is a poly of degree less than degree of denominator. These always have a limit of zero as $x \to \pm\infty$.

Example

$$\lim_{x \to -\infty} \frac{4x-1}{\sqrt{x^2+2}} = \lim_{x \to -\infty} \frac{x(4-\frac{1}{x})}{\sqrt{x^2+2}}$$

$$= \lim_{x \to -\infty} \frac{4-\frac{1}{x}}{\left(\frac{1}{x}\right)\sqrt{x^2+2}}$$

$$= \lim_{x \to -\infty} \frac{4-\frac{1}{x}}{-\frac{1}{|x|}\sqrt{x^2+2}}$$

$$= \lim_{x \to -\infty} -\frac{(4-1/x)}{\sqrt{1+\frac{2}{x^2}}}$$

$$= -4$$

Horizontal and Vertical Asymptotes

Def

A horizontal line $y=b$ is a horizontal asymptote of $y=f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

A vertical line $x=a$ is a vertical asymptote of $y=f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

Example

Find any vertical or horizontal asymptotes of $f(x) = \dfrac{4x-5}{3x+2}$

Solution:

a) Horizontal asymptote(s):

$$\lim_{x\to\infty} \frac{4x-5}{3x+2} = \lim_{x\to\infty} \frac{x(4-5/x)}{x(3+2/x)}$$

$$= \lim_{x\to\infty} \frac{4-5/x}{3+2/x}$$

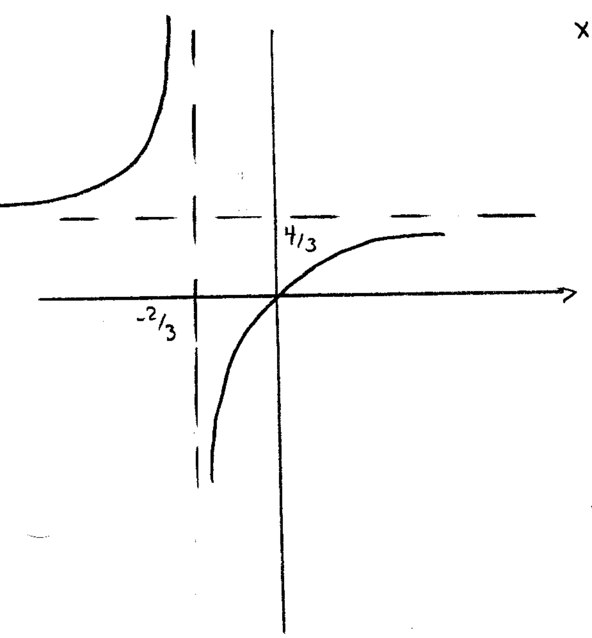
$$= \frac{4}{3}$$

Note $\displaystyle\lim_{x\to-\infty} \frac{4x-5}{3x+2} = \frac{4}{3}$ as well.

Thus $y = \frac{4}{3}$ is a horizontal asymptote for $x \to \pm\infty$.

b) Vertical asymptote(s): The denominator is zero at $x = -\frac{2}{3}$, so:

$$\lim_{x\to-\frac{2}{3}^-} \frac{4x-5}{3x+2} = \lim_{x\to-\frac{2}{3}^-} \frac{4(\frac{2}{3})-5}{3(x+\frac{2}{3})}$$



$$= -\frac{7}{9} \underbrace{\lim_{x\to-\frac{2}{3}^-} \frac{1}{x+\frac{2}{3}}}_{-\infty}$$

$$= +\infty$$

$$\lim_{x\to-\frac{2}{3}^+} \frac{4x-5}{3x+2} = -\infty$$

Oblique Asymptotes

A rational function is one that consists of a polynomial in the numerater and another polynomial in the denominater.

If the degree of the polynomial in the numerater is one degree larger than the degree of the numerater, there will be a straight line (linear) asymptote, but it will in general be neither vertical or horizontal.

Example

Graph

$$f(x) = \frac{(x-1)^3}{x^2}$$

$$= \frac{x^3 - 3x^2 + 3x + 1}{x^2}$$

$$= \underbrace{(x - 3)}_{\text{straight line of slope 1 and y intercept -3}} + \left(\frac{3}{x} + \frac{1}{x^2}\right)$$

As $|x|$ gets large, $\frac{1}{x} + \frac{1}{x^2}$ gets smaller and smaller. On the otherhand $|x-3|$ gets larger and larger. So the first term dominates as $|x|$ gets large. Thus we can say

$$f(x) \approx x - 3 \qquad \text{for } |x| >> 1$$

This is a straight line of slope 1 and y intercept -3. This is the oblique asymptote as $x \to \pm\infty$.

Next, what happens at $x=0$ where the denominator vanishes?

$$\lim_{x\to 0^-} \frac{(x-1)^3}{x^2} = \lim_{x\to 0^-} \frac{(-1)^3}{x^2}$$

$$= -\infty$$

and

$$\lim_{x\to 0^+} \frac{(x-1)^3}{x^2} = -\infty$$

as well. Also note that $f(x)=0$ when $x=1$.
What about max and min?

$$f'(x) = \frac{d}{dx} \frac{(x-1)^3}{x^2}$$

$$= \frac{x^2\, 3(x-1)^2 - (x-1)^3\, 2x}{x^4}$$

$$= \frac{x(x-1)^2\left[3x - 2(x-1)\right]}{x^4}$$

$$= \frac{(x-1)^2(x+2)}{x^4}$$

So $f'(x)=0$ at $x=-2$ and $x=1$. Note that

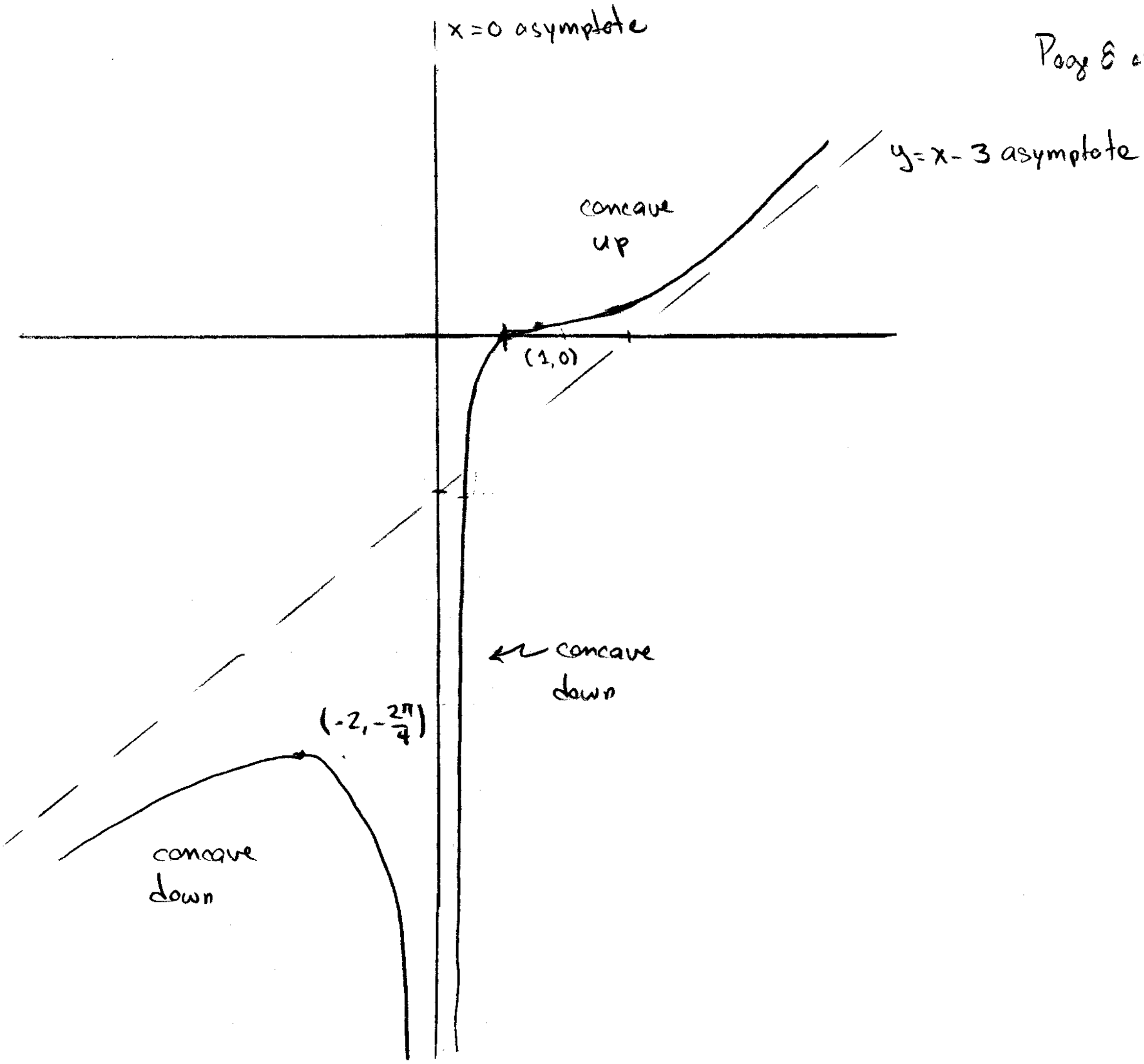
$f'(x) < 0$ for $x < -2$
$f'(x) > 0$ for $-2 < x < 0$
$f'(x) > 0$ for $x > 0$

From the 1st derivative test, $f(-2) = -\frac{27}{4}$ is a local max. We can also show that

$$f''(x) = \frac{6(x-1)}{x^4}$$

$$\Rightarrow f''(1) = 0$$

Hence $x=1$ is an inflection point.



Example

Graph $f(x) = x^2 + \frac{2}{x}$

Soln:

a) $\lim_{x \to \pm\infty} x^2 + \frac{2}{x}$

b) $\lim_{x \to 0^-} x^2 + \frac{2}{x} = -\infty$

c) $\lim_{x \to 0^+} x^2 + \frac{2}{x} = +\infty$

So no horizontal asymptotes, but a vertical asymptote at $x = 0$.

d) $\frac{d}{dx}(x^2 + \frac{2}{x}) = 2x - \frac{2}{x^2}$

$$= x - \frac{1}{x^2}$$

$$= \frac{x^3 - 1}{x^2}$$

Thus $f'(x) = 0$ at $x = 1$. Also,

e) $\frac{d^2}{dx^2}(x^2 + \frac{2}{x}) = \frac{d}{dx}(x - \frac{1}{x^2})$

$$= 1 + \frac{2}{x^3}$$

Then $f''(1) > 0 \Rightarrow$ min at $x = 1$, Further

$$1 + \frac{2}{x^3} = 0$$

$$\Rightarrow x^3 = -2$$

$$\Rightarrow x = -2^{1/3}$$

So $f''(-2^{1/3}) = 0$. Note that $f''(x) > 0$ for $x < -2^{1/3}$ and $f''(x) < 0$ for $-2^{1/3} < x < 0$. So $x = -2^{1/3}$ is an inflection point.

f) $f(x) = 0 \Rightarrow x^2 + \frac{2}{x} = 0$

$$\Rightarrow x^3 + 2 = 0$$

$$\Rightarrow x = -2^{1/3}$$

is a root.

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