APPM 1350: Section 4.2: Differential Equations

A large percentage of the laws of nature are expressed as relations between derivatives and some function. For example, Newton's 2nd law of motion relates the derivative of the momentum of a body to the force on the body.

$$\frac{dp}{dt} = F(t)$$

[Note that p is a function of time]

where F = force

$p = mv$ = momentum

m = mass

v = velocity

Equations that relate derivatives to functions are called differential equations. When we augment the differential equation with knowledge of a specific value of the unknown (for example $p(t_0) = p_0$ above) then the problem of solving the differential equation is called an initial value problem.

To solve

$$\frac{dp}{dt} = F(t)$$

we integrate both sides with respect to t:

$$\int \frac{dp}{dt}\, dt = \int F(t)\, dt$$

The left-hand integral produces the antiderivative of $\frac{dp}{dt}$ which is just $p(t)$. Hence:

$$p(t) = \int F(t)\, dt$$

Example

Assume we throw a ball of mass m vertically upward with an initial velocity of v_0. What is the velocity as a function of time? Assume the acceleration of gravity is a constant g.

Solution:

Assume we are starting at $z=0$ and $z>0$ is upward. The force of gravity on the ball is:

$$F = -mg$$

where the negative is because gravity wants to push the ball downward ($z<0$). From Newton's 2nd law of motion:

$$\frac{dp}{dt} = F$$

$$\Rightarrow \frac{dp}{dt} = -mg$$

$$\Rightarrow \int \frac{dp}{dt}\,dt = \int (-mg)\,dt$$

$$\Rightarrow p(t) = -mg \int dt \qquad \text{(since } m \text{ and } g \text{ are constants)}$$

$$= -mgt + c$$

Since $p = mv$ and m is constant:

$$mv(t) = -mgt + c$$

$$\Rightarrow v(t) = -gt + \frac{c}{m}$$

$$\Rightarrow v(t) = -gt + c_2 \qquad c_2 = \frac{c}{m} \qquad (1)$$

Assume we threw the ball at time $t=0$. Then our initial condition was

$$V(t=0) = V_0$$

Plug this into (1):

$$V(0) = V_0 = -g(0) + c_2$$

$$\Rightarrow c_2 = V_0$$

Hence (1) becomes:

$$V(t) = -gt + V_0$$

Example

An electron is moving in a sinusoidal electric field that is exerting a force $F(t) = \cos \alpha t$ on it. Assume the electron motion is along the x axis and that the velocity and position of the electron at $t=0$ are $V(0) = V_0$ and $X(0) = X_0$. Find the velocity and position of the electron as a function of time.

Solution.

The mass of the electron (m) is a constant. So we can write Newton's law as:

$$F = m\frac{dv}{dt}$$

For $F = \cos t$, we thus have

$$m\frac{dv}{dt} = \cos \alpha t$$

$$\Rightarrow \frac{dv}{dt} = \frac{1}{m}\cos\alpha t$$

$$\Rightarrow \int \frac{dv}{dt}\,dt = \int \frac{1}{m}\cos\alpha t\,dt$$

$$\Rightarrow v(t) = \frac{1}{m}\int \cos\alpha t\,dt$$

$$= -\frac{1}{m}\,\frac{\sin\alpha t}{\alpha} + c$$

We know that $V(0) = V_0$, hence

$$V_0 = -\frac{1}{\alpha m}\,\cancel{\sin\alpha(0)}^{\,0} + c$$

$$\Rightarrow c = V_0$$

Hence the velocity as a function of time is

$$V(t) = -\frac{1}{\alpha m}\sin\alpha t + V_0$$

For the position

$$V(t) = \frac{dx}{dt} = -\frac{1}{\alpha m}\sin\alpha t + V_0$$

$$\Rightarrow \int \frac{dx}{dt}\,dt = \int\left(-\frac{1}{\alpha m}\sin\alpha t + V_0\right)dt$$

$$\Rightarrow x(t) = -\frac{1}{\alpha m}\int \sin\alpha t\,dt + V_0\int dt$$

$$= \frac{1}{\alpha m}\,\frac{\cos\alpha t}{\alpha} + V_0 t + c$$

Since $X(0) = X_0$,

$$X_0 = \frac{1}{\alpha^2 m}\cos\alpha\cdot 0 + V_0\cdot 0 + C$$

$$\Rightarrow \quad C = x_0 - \frac{1}{\alpha^2 m}$$

Hence the position as a function of time is:

$$x(t) = \frac{1}{\alpha^2 m}\cos\alpha t + v_0 t + x_0 - \frac{1}{\alpha^2 m}$$

As an aside, this technique of solving a differential equation by integrating in this fashion is called the technique of separation of variables.

Example
(#18, sec 4.2)
Solve the initial value problem

$$\frac{d^2 s}{dt^2} = \frac{3t}{8} \qquad \left.\frac{ds}{dt}\right|_{t=4} = 3 \qquad s(4) = 4$$

Solution:

$$\frac{d^2 s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right)$$

so

$$\frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{3t}{8}$$

$$\Rightarrow \int \frac{d}{dt}\left(\frac{ds}{dt}\right) = \int \frac{3t}{8}\,dt$$

$$\Rightarrow \frac{ds}{dt} = \frac{3t^2}{16} + c$$

Since $\frac{ds}{dt} = 3$ at $t = 4$

$$3 = \frac{3}{16}(4)^2 + c$$

$$\Rightarrow C = 0$$

Thus:

$$\frac{ds}{dt} = \frac{3t^2}{16}$$

$$\Rightarrow \int \frac{ds}{dt}\,dt = \int \frac{3t^2}{16}\,dt$$

$$\Rightarrow s(t) = \frac{t^3}{16} + C$$

Since $S(4) = 4$:

$$4 = \frac{(4)^3}{16} + C$$

$$\Rightarrow C = 0$$

and thus:

$$s(t) = \frac{t^3}{16}$$

Families of Curves

Consider

$$\frac{dF}{dt} = F(t)$$

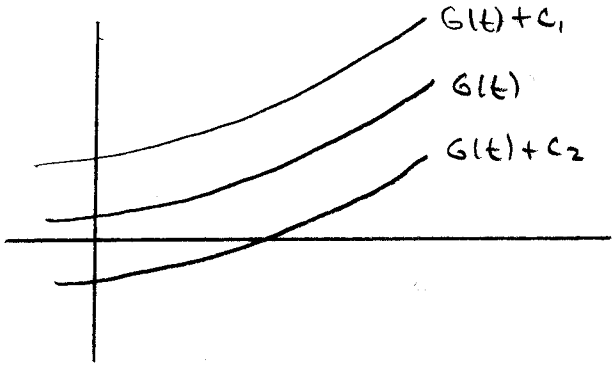
$$\Rightarrow \int \frac{df}{dt}\,dt = \int F(t)\,dt$$

$$\Rightarrow f(t) = \int F(t)\,dt$$

$$= G(t) + C$$

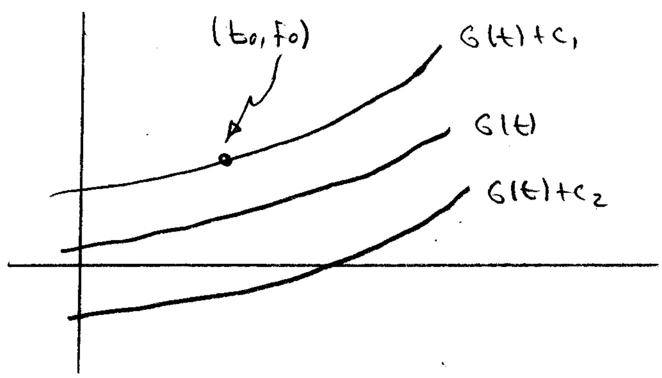
$\boxed{\text{Assume } G(t) \text{ is an antiderivative of } F}$

Since C is a constent, $f(t) = G(t) + C$ represents a family of curves that differ by a constent.



and so forth.

The initial value $f(t_0) = f_0$ fixes the solution of the initial value problem to lie on one of the otherwise infinite family of curves. So if the point (t_0, f_0) is a shown below then the solution of



interest to us is the one that passes through (t_0, f_0). In our case that is the curve $G(t) + C_1$. The constent C_1 in this case would be $f_0 - G(t_0)$.

Sketching Solution Curves

Sometimes it is difficult to solve a differential equation either because the integral is too difficult or can't be represented by simple functions. However since we have the derivative of the function we can still at least sketch the solution.

Example

(#42, sec 4.2)

Sketch the solutions to the initial value problem

$$\frac{dy}{dx} = \sqrt{1+x^4} \qquad y(0) = 1$$

Solution:

The domain of this function is $\{x : x \in \mathbb{R}\}$. There are no critical points where $y'(x) = 0$ or $y'(x)$ doesn't exist. So there are no local minima or maxima. What about inflection points?

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\sqrt{1+x^4}$$

$$= \frac{4x^3}{2\sqrt{1+x^4}}$$

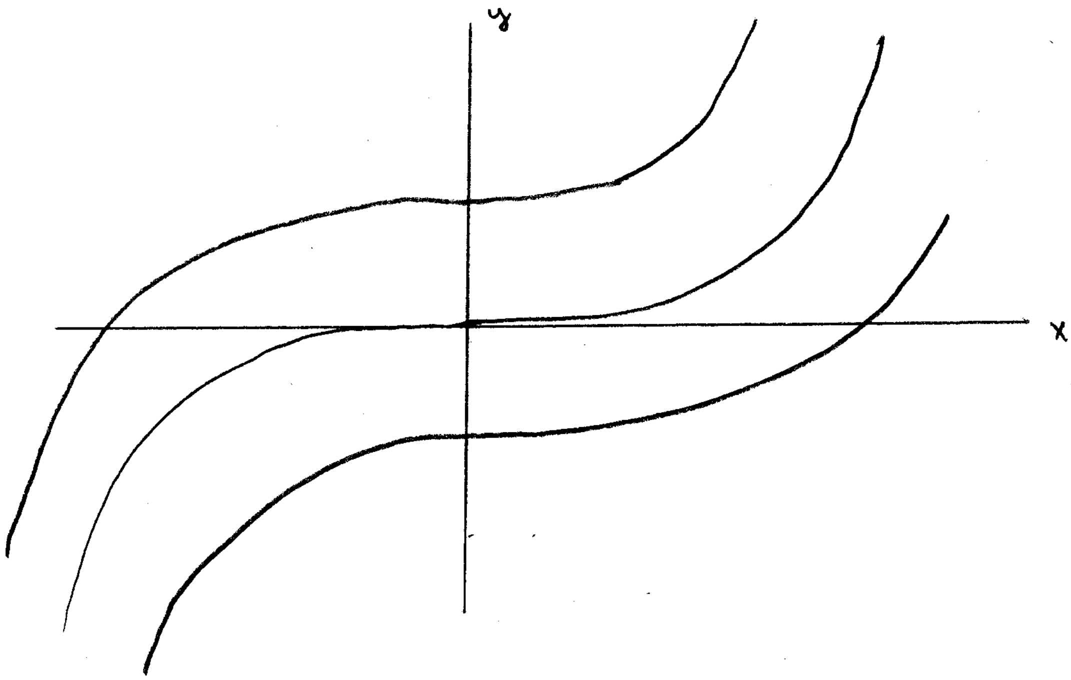
$$= \frac{2x^3}{\sqrt{1+x^4}}$$

Thus $\frac{d^2y}{dx^2} = 0$ at $x = 0$ $\Rightarrow$ inflection point at $x = 0$. Further,

$$\frac{d^2y}{dx^2} < 0 \text{ for } x < 0 \Rightarrow \text{concave down}$$

$$\frac{d^2y}{dx^2} > 0 \text{ for } x > 0 \Rightarrow \text{concave up}$$

So a <u>family of solutions</u> would look something like:



all of which differ by a constant. We are interested in only one of these curves, the one where $y(0) = 1$. Thus a sketch of the solution of interest to us is: