APPM 1350: Section 4.8: Substitution in Definite Integrals

In the previous section we noted that if $f(x)$ is continuous on $[a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is <u>any</u> antiderivative of $f(x)$, hence

$$F(x) = \int f(x)\,dx$$

Definite integration thus reduces to computing the indefinite integral and then subtracting that function evaluated at the endpoints.

<u>Example</u>

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi$$

$$= -(-1) + 1$$

$$= 2$$

Note;

$$\cos x\Big|_0^\pi = \cos\pi - \cos 0$$

means evaluate $\cos x$ at the endpoints and subtract. More generally

$$\boxed{F(x)\Big|_a^b = F(b) - F(a)}$$

So what about more complicated integrals? In particular, suppose we need substitution to evaluate the indefinite integral? Then one approach is do the substitution, change back to the original variable, and then evaluate the integral.

<u>Example</u>

(#4, sec 4.8) Evaluate $\int_0^{\pi} 3\cos^2 x \sin x \, dx$

Solution:

$$F(x) = \int 3\cos^2 x \sin x \, dx$$

$$= 3\int \cos^2 x \sin x \, dx$$

Let $u = \cos x$

$\Rightarrow du = -\sin x \, dx$

$$= -3\int u^2 du$$

$$= -u^3 + C$$

$$= -\cos^3 x + C$$

$\leftarrow$ substitute back for x

Hence

$$\int_0^{\pi} 3\cos^2 x \sin x \, dx = F(\ ) - F(0)$$

$$= -\cos^3 \pi + \cos^3 0$$

$$= 1 + 1$$

$$= 2$$

There is another, often more convient, way. When you make the substitution(s), change the limits of integration to be consistent with the new variable of integration. So using the same example as before

<u>Example</u>
(#4, sec 4.8) Evaluate $\displaystyle\int_0^{\pi} 3\cos^2 x \sin x \, dx$

Solution:

$$\int_0^{\pi} 3\cos^2 x \sin x \, dx = 3\int_0^{\pi} \cos^2 x \sin x \, dx$$

Let $u = \cos x$
$\Rightarrow du = -\sin x \, dx$
Further:
$\cos \pi = -1$
$\cos 0 = 1$

$$= -3\int_1^{-1} u^2 \, du$$

$$= 3\int_{-1}^{1} u^2 \, du \qquad \text{(invert the limits of integration)}$$

$$= u^3 \Big|_{-1}^{1}$$

$$= 1 - (-1)$$

$$= 2$$

So where we changed variables of integration we also changed the limits of integration using our substitution.

Example
(#16, sec 4.8) Evaluate $\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$

Solution:

$$\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2} = \frac{1}{2}\int_1^4 \frac{dy}{\sqrt{y}(1+\sqrt{y})^2}$$

Let $u = 1+\sqrt{y}$

$du = \frac{1}{2\sqrt{y}}\,dy$

and for $y=1$: $u=2$

for $y=4$: $u=3$

$$= \int_2^3 \frac{du}{u^2}$$

$$= -\frac{1}{u}\Big|_2^3$$

Note I dropped the arbitrary constant that accompanies the indef. integral. This is because we can use any antiderivative in the definite integral, so I set the constant to zero. If it was nonzero it won't matter, it will cancel when we do the subtraction at the endpoints.

$$= -\frac{1}{3} + \frac{1}{2}$$

$$= \frac{1}{6}$$

Finding Areas

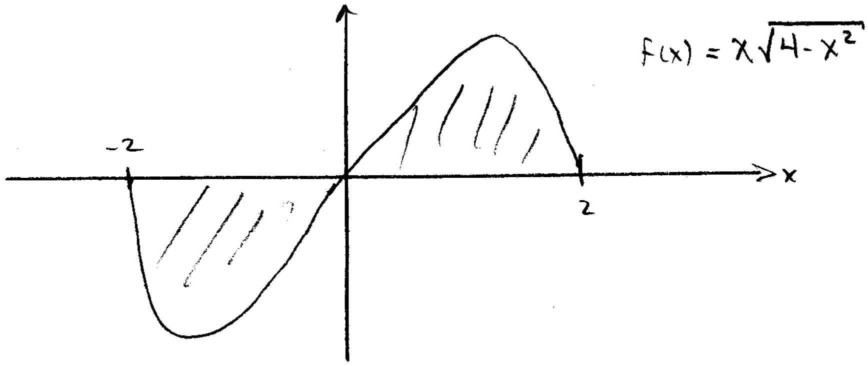
The total area under a curve between $x=a$ and $x=b$ is:

$$\text{Area} = \int_a^b |f(x)|\,dx$$

The absolute value is there because the definite integral of a negative function is negative, but we want the "area" to be positive.

Example

(#25, sec 4.8) Find the total area of the shaded region below:



Solution:

$$\text{Area} = \int_{-2}^{2} |x\sqrt{4-x^2}|\,dx$$

$$= \int_{-2}^{0} |x\sqrt{4-x^2}|\,dx + \int_{0}^{2} |x\sqrt{4-x^2}|\,dx$$

$$= -\int_{-2}^{0} x\sqrt{4-x^2}\,dx + \int_{0}^{2} x\sqrt{4-x^2}\,dx$$

since $|x| = -x$ if $x < 0$

$$= \frac{1}{2}\int_{0}^{4} u^{1/2}\,du - \frac{1}{2}\int_{4}^{2} u^{1/2}\,du$$

Let $u = 4-x^2$

$du = -2x\,dx$

For $x = -2$: $u = 0$

$x = 0$: $u = 4$

$x = 2$: $u = 0$

$$= \left.\frac{u^{3/2}}{3}\right|_{0}^{4} - \left.\frac{u^{3/2}}{3}\right|_{4}^{0}$$

$$= \left(\frac{8}{3} - 0\right) - \left(0 - \frac{8}{3}\right)$$

$$= \frac{16}{3}$$

Some Interesting Theory

(#34, sec 4.8)

By using a substitution, prove that for all positive numbers x and y;

$$\int_x^{xy} \frac{1}{t}\,dt = \int_1^{y} \frac{1}{t}\,dt$$

Solution:

$$\int_x^{xy} \frac{1}{t}\,dt = \int_1^{y} \left(\frac{1}{ux}\right)(x\,du)$$

Let $u = \frac{t}{x}$

$\Rightarrow du = \frac{1}{x}\,dt$

and if $t = x \Rightarrow u = 1$

$t = xy \Rightarrow u = y$

$$= \int_1^{y} \frac{du}{u}$$

$$= \int_1^{z} \frac{dt}{t}$$

The last step is not a substitution per se, rather we just changed the dummy variable of integration. Note that the restriction that x and y both be positive is to guarantee that the integrand in continuous in the interval. Otherwise all bets are off for the time being.

This example is of no great general importance, but it does illustrate the kinds of manipulations possible with definite integrals.