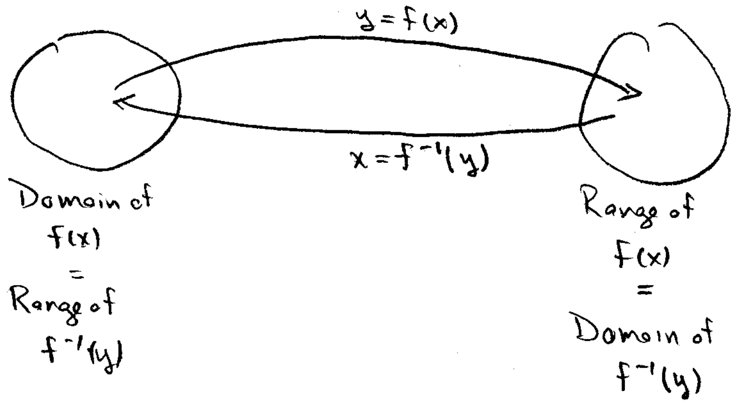
# APPM 1350: Section 6.1: Inverse Functions and Their Derivatives

Given a function $f(x)$ that maps $x$ from the domain $D$ to the range $R$. So when we write $y=f(x)$ we are saying that $f$ maps $x$ into $y$. The <u>inverse</u> of $f(x)$ is a function, if it exists, that maps $y$ back into $x$. We would write $x=f^{-1}(y)$. The domain of $f^{-1}(y)$ is the range of $f(x)$ and the range of $f^{-1}(y)$ is the domain of $f(x)$. Hence




A example of a function and its inverse is:

$$f(x)=x^3 \qquad f^{-1}(x)=\sqrt[3]{x}$$

$f$ maps $x$ into $y=x^3$ and $f^{-1}$ maps $y=x^3$ into $x=\sqrt[3]{y}=x$. Hence

> Functions $f$ and $g$ are an inverse pair if and only if
>
> $$f[g(x)] = g[f(x)] = x$$
>
> In this case $g=f^{-1}$ and $f=g^{-1}$

Note that in the example and theorem we wrote $f(x)$ and $f^{-1}(x)$ rather than $f^{-1}(y)$. The variable is just a place holder, you can write $f^{-1}(x)$, $f^{-1}(y)$, $f^{-1}(\text{Cortez})$. Dummy variables are just that and are completely arbitrary. The functional form of $f^{-1}(\cdot)$ is all that matters.

Now, when do inverses of functions exist? They exist whenever the function is <u>one-to-one</u>.

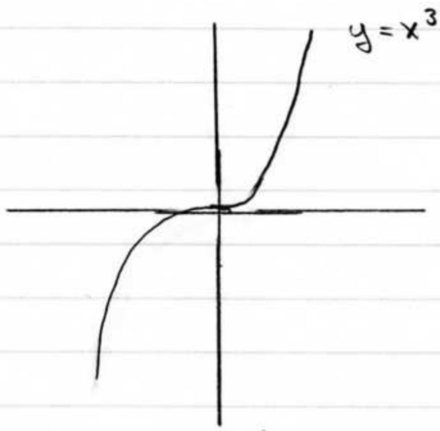
> **Def:** A function $f(x)$ is <u>one-to-one</u> on a domain $D$ if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

A graphical interpretation of this is:

> <u>The Horizontal Line Test</u> A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

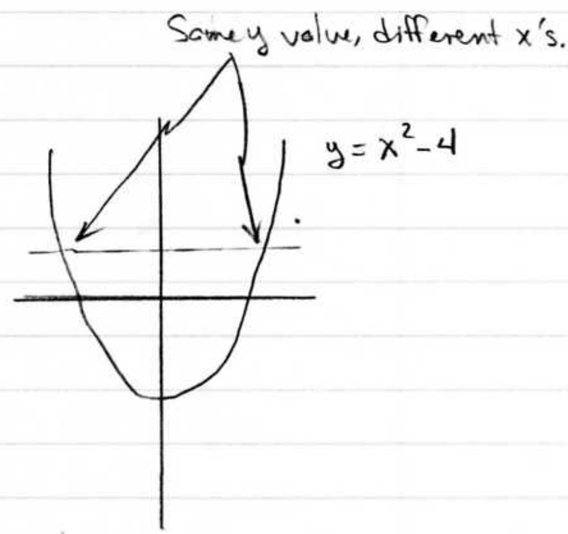
So for any value of $x$ in the domain of $f(x)$ there is only one $y$ value <u>and</u> for each $y$ in the range of $f(x)$ there is only one $x$ value associated with it.

<u>Examples</u>




1-to-1

All horizontal lines intersect at most once.




<u>Not</u> 1-to-1

Any line $y > -4$ intersects twice

So, back to inverses:

| |
|---|
| A function has an inverse if and only if it is one-to-one |

## Example

$$f(x) = \sqrt{x}$$

The domain is $[0,\infty)$ and the range is $[0,\infty)$ as well. The inverse is the function

$$g(x) = x^2$$

Thus:

$$f[g(x)] = \sqrt{x^2} = x$$

and

$$g[f(x)] = (\sqrt{x})^2 = x$$

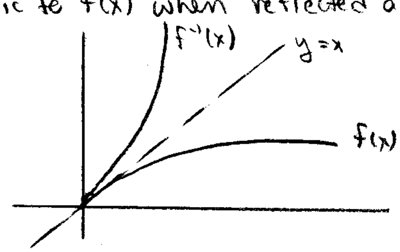
Hence $g(x) = f^{-1}(x)$ and $f(x) = g^{-1}(x)$.

| |
|---|
| Note: Whenever $f[g(x)] = g[f(x)] = x$ (or $f \circ g = g \circ f = x$), we say that $f \circ g = g \circ f$ is the <u>identity function</u>. The identity function assigns $x$ to itself. |

## Graphing Inverses

The graph of $f^{-1}(x)$ is symmetric to $f(x)$ when reflected across the line $x=y$.

For example:



## Finding Inverses

Given $y = f(x)$, we find $f^{-1}(x)$ by:

1. Solve $y = f(x)$ for $x$ in terms of $y$. So we have $x = g(y)$

2. The inverse $f^{-1}(x)$ written in terms of $x$ is just
$$f^{-1}(x) = g(x)$$

### Example

(#20, sec 6.1)

$$f(x) = x^4 \qquad x \geq 0$$

Note that the domain of $f$ is $[0,\infty)$ and the range is also $[0,\infty)$. Solving for $x$ in terms of $y$:

$$y = x^4$$

$$\Rightarrow x = \underbrace{y^{1/4}}_{f^{-1}(y)}$$

Hence

$$f^{-1}(x) = x^{1/4}$$

(just swap the $x$ and $y$'s). Note that the domain of $f^{-1}(x)$ is $[0,\infty)$ and the range is also $[0,\infty)$. So the domain of $f$ is the range of $f^{-1}$ and the range of $f$ is the domain of $f^{-1}$. Note also that

$$f \circ f^{-1} = (x^{1/4})^4 = x$$

$$f^{-1} \circ f = (x^4)^{1/4} = x$$

as required by the definition of inverse functions.

Examde

$$f(x) = \frac{1+x}{1-x}$$

Note that the domain of f is $D_f = \{\mathbb{R} : x \neq 1\}$ and the range is $R_f = \{\mathbb{R} : y \neq -1\}$. The range excludes $y = -1$ because this is a horizontal asymptote of f, hence it never actually achieves the value $y = -1$ for any finite x.

To find the inverse of $f(x)$, write $y = f(x)$ and solve for x in terms of y:

$$y = \frac{1+x}{1-x}$$

$$\Rightarrow (1-x)y = 1+x$$

$$\Rightarrow x(-y-1) = 1-y$$

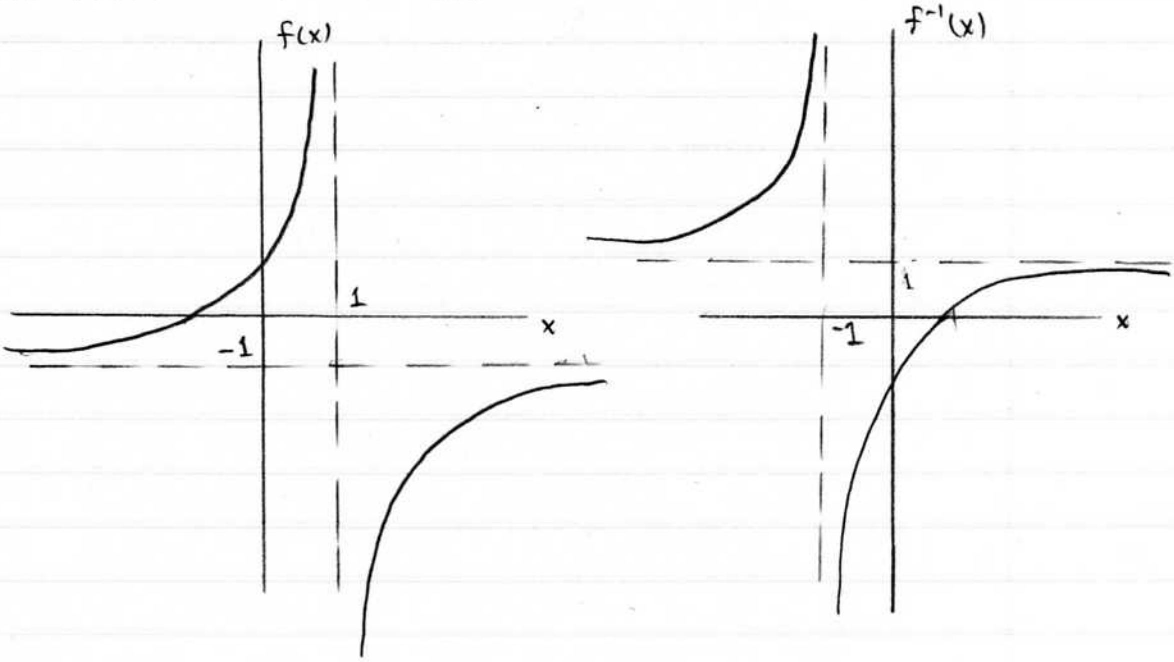
$$\Rightarrow x = \underbrace{\frac{y-1}{y+1}}_{f^{-1}(y)}$$

Hence:

$$f^{-1}(x) = \frac{x-1}{x+1}$$

Note, the domain of $f^{-1}$ is $D_{f^{-1}} = \{\mathbb{R}, x \neq -1\}$ and the range is $R_{f^{-1}} = \{\mathbb{R} : y \neq 1\}$ where $y = 1$ is excluded because it is a horizontal asymptote of $f^{-1}(x)$.

Note the plots of $f(x)$ and $f^{-1}(x)$:




## Derivatives of Inverses of Differentiable Functions

Suppose $f(x)$ is differentiable and it has an inverse. Then writing $y = f(x)$ we have at the point $x = a$:

$$\frac{dy}{dx} = f'(x)$$

or

$$dy = f'(x)\,dx \qquad (x = a)$$

So long as $f'(x) \neq 0$, we can continue to manipulate the differentials algebraically. Hence:

$$1 = f'(x)\frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{f'(x)} \qquad (f'(x) \neq 0)$$

But $x = f^{-1}(y)$, so

$$\frac{d}{dy} f^{-1}(y) \Bigg|_{y=f(a)} = \frac{1}{\frac{d}{dx} f(x)} \Bigg|_{x=a}$$

We can also write this in terms of $f^{-1}(x)$ instead of $f^{-1}(y)$ provided we are careful to note that the <u>two derivatives are evaluated at different x.</u> Specifically:

<u>The Derivative Rule for Inverses</u> If $f$ is differentiable at every point of an interval $I$ and $df/dx$ is never zero on $I$, then $f^{-1}$ is differentiable at every point of the interval $f(I)$. The value of $df^{-1}/dx$ at any particular point $f(a)$ is the <u>reciprocal of the value</u> of $df/dx$ at $a$:

$$\left(\frac{df^{-1}}{dx}\right)_{x=f(a)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=a}}$$

In short;

$$(f^{-1})' = \frac{1}{f'}$$

Another way this is often written is:

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

Example

$$f(x) = 2x^2 \qquad x \geq 0$$

$$\Rightarrow y = 2x^2$$

$$\Rightarrow x = \sqrt{\frac{y}{2}} \qquad \Rightarrow f^{-1}(x) = \sqrt{\frac{x}{2}}$$

At the point $x=5$, $y=50$. Then from our theorem,

$$\left.\frac{d}{dx} f^{-1}(x)\right|_{x=50} = \left.\frac{1}{\frac{d}{dx} f(x)}\right|_{x=5} = \left.\frac{1}{4x}\right|_{x=5} = \frac{1}{20}$$

To check that, note that

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{\sqrt{2}} \frac{d}{dx} \sqrt{x}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2\sqrt{x}} \qquad (1)$$

At $x = f(5) = 50$ we have

$$\frac{d}{dx} f'(x) = \frac{1}{2\sqrt{2}\sqrt{50}}$$

$$= \frac{1}{\sqrt{8 \cdot 50}}$$

$$= \frac{1}{20}$$

However, more generally, given $y = 2x^2 \quad (x \geq 0)$, we have:

$$\underbrace{\frac{dx}{dy}}_{(f^{-1})'(y)} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{1}{\frac{d}{dx} 2x^2}$$

$$= \frac{1}{4x}$$

So

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{4x}$$

Since $x = \sqrt{\frac{y}{2}}$, upon substitution we have

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{4\sqrt{\frac{y}{2}}}$$

$$= \frac{1}{2\sqrt{2y}}$$

Swapping x and y:

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{2\sqrt{2x}}$$

which is what we got in equation (1) on the previous page.