APPM 1350: Section 6.5: Growth and Decay

I introduced the concept of exponential decay during class a week ago. We now explore it in more detail. A significant number of physical and biological systems that vary with time (or some other parameter) are such that the rate of change of some value is proportional to the value. Hence these systems are modeled by the following initial value problem:

$$\boxed{\begin{aligned} \frac{dy}{dt} &= ky \\ y(0) &= y_0 \end{aligned}}$$

k is called the <u>constant of proportionality</u>. Note that when $y>0$ and $k>0$ then $dy/dt>0$ whereas when $y>0$ and $k<0$ then $dy/dt<0$. Typically $k>0$ implies growth and $k<0$ implies decay of some value y. Now solve the IVP:

$$\frac{dy}{dt} = ky \qquad (k = \text{constant})$$

$$\Rightarrow dy = ky\,dt$$

$$\Rightarrow \frac{dy}{y} = k\,dt$$

$$\Rightarrow \int \frac{dy}{y} = \int k\,dt$$

$$\Rightarrow \ln|y| = kt + c$$

$$\Rightarrow |y| = e^{kt+c}$$

$$= e^c e^{kt}$$

or more generally;

$$y = \pm e^{c} e^{kt}$$

$$= Ae^{kt} \qquad A = \pm e^{c} = \text{constant}$$

Since $y(0) = y_0$, then the solution to the IVP is:

$$y(t) = y_0 e^{kt}$$

Hence:

<u>The Law of Exponential Change</u>

$$y = y_0 e^{kt}$$

Growth: $k > 0$ Decay: $k < 0$

k is the <u>rate constant</u> of the equation, and $y_0 = y(0)$

Note that there is another way to derive this solution to the IVP. Go back to

$$\int \frac{dy}{y} = \int k\,dt$$

Change the integral on the left into a definite integral from y_0 to $y(t)$ and the integral on the right into a definite integral from $t = 0$ to t.

Thus:

$$\int_{y_0}^{y(t)} \frac{dy}{y} = k\int_0^t dt$$

or using different dummy variables of integration (not necessary to do but often clarifies what is what):

$$\int_{y_0}^{y(t)} \frac{d\alpha}{\alpha} = k \int_0^t d\beta$$

Integrating:

$$\ln \alpha \Big|_{y_0}^{y(t)} = k\beta \Big|_0^t$$

$$\Rightarrow \ln|y(t)| - \ln|y_0| = k(t-0)$$

$$\Rightarrow \ln\left|\frac{y(t)}{y_0}\right| = kt$$

$$\Rightarrow \left|\frac{y(t)}{y_0}\right| = e^{kt}$$

Now I use a trick. Since $y(0) = y_0$, the ratio y/y_0 is positive for $t=0$. So $y(t)$ starts out with the same sign as y_0. Does it ever change to a different sign? The answer is no. If $y_0 < 0$ and $k < 0$ then $y(t)$ will become more negative as time increases, so $y(t)/y_0 > 0$. If $k > 0$ then $y(t)$ will indeed become less and less negative with time. But if somehow $y(t)$ were to become zero at some t then $dy/dt = 0$ and $y(t)$ would stop changing at that point (any value y for which $y' = 0$ is called an equilibrium solution of this kind of differential equation). So irrospective of k, $y(t)/y_0 \geq 0$.

Hence

$$\left|\frac{y(t)}{y_0}\right| = \frac{y(t)}{y_0}$$

and so the solution to the initial value problem is

$$y(t) = y_0 e^{kt}$$

A similiar argument applies where $y_0 > 0$.

Either approach is an acceptable way to solve the initial value problem. Mathematics classes tend to favor the first approach, physics classes tend to favor the second approach.

We now explore some specific examples of physical or biological systems that obey the law of exponential change.

Population Growth

If you have a large population of people, some percentage of them will be reproducing at any time. The more people the more who are reproducing (given adequate food, etc...). Mathematically we can model this as

$$\frac{dy}{dt} = ky$$

where $k > 0$ since we are considering only births (not deaths). If there are y_0 people at some time we call zero, then $y(0) = y_0$ and the population at some time $t > 0$ will be:

$$y(t) = y_0 e^{kt}$$

Example

(#7, see 6.5) Suppose that the bacteria in a colony can grow unchecked, by the law of exponential change. The colony starts with 1 bacterium and doubles every half hour. How many bacteria will the colony contain at the end of 24 hours?

Solution:

Let B = number of bacteria. We have $B(0) = 1$ and assume a rate constant $k > 0$ (we don't know its value yet). Then

$$\frac{dB}{dt} = kB$$

by the law of exponential change and

$$B(t) = B_0 e^{kt} = e^{kt} \quad \text{since } B(0) = B_0 = 1$$

We also know that $B(\frac{1}{2}) = 2B(0) = 2B_0$ since B doubles every half hour. So:

$$2B_0 = B_0 e^{k(\frac{1}{2})}$$

$$\Rightarrow 2 = e^{k/2}$$

$$\Rightarrow \ln 2 = \frac{k}{2}$$

$$\Rightarrow k = 2\ln 2 \ \frac{1}{hr}$$

Thus $B(t)$ is:

$$B(t) = e^{(2\ln 2)t}$$

and

$$B(24) = e^{(2\ln 2)(24)} = 4^{24} \approx 2.8 \times 10^{14} \text{ bacteria}$$

Continuously Compounded Interest

If you get a continuous interest rate r on some amount of money A, then by definition

$$\frac{dA}{dt} = rA$$

(note both r and A are positive). So if $A(0) = A_0$ we have

$$A(t) = A_0 e^{rt}$$

Radioactivity

When an atom decays through fission it converts some percentage of its energy into energy in the form of a gamma ray while also emitting one or more neutrons. This results in the atom splitting into two atoms that are generally more stable. This is the process of radioactive decay. Given a large number N of radioactive atoms (i.e., atoms that can undergo fission) the rate at which the atoms decay (i.e., fission) can be shown to be proportional to N.

$$\frac{dN}{dt} = -kN$$

Here $k > 0$ and the negative sign indicates N is decreasing with time. Thus:

$$N(t) = N_0 e^{-kN}$$

where $N(0) = N_0$. The half-life of a radioactive element is the time required for half the atoms to decay; i.e., the time $t_{1/2}$ such that $N(t_{1/2}) = \frac{1}{2}N_0$. Thus:

$$\frac{N_0}{2} = N_0 e^{-kt_{1/2}}$$

$$\Rightarrow \frac{1}{2} = e^{-kt_{1/2}}$$

$$\Rightarrow t_{1/2} = -\frac{1}{k}\ln\left(\frac{1}{2}\right) = \frac{\ln 2}{k}$$

So:

$$\boxed{\text{Half-life} = \frac{\ln 2}{k}}$$

Many chemical and biological reactions obey a similiar behavior, for example the rate at which many drugs are metebolized and purged by your body is proportional to the amount of drug present. Consequently we can talk about the half-life of a drug in your body in the same way as we talk about the half-life of a radioactive substance.

Example

(#18, see 6.5) The half-life of polonium is 139 days, but your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives has disintegrated. For about how many days after the sample arrives will you be able to use the polonium?

Solution:

$$N = N_0 e^{-kt}$$

and

$$\frac{\ln 2}{k} = 139 \text{ days}$$

$$\Rightarrow k = \frac{\ln 2}{139}$$

Hence:

$$N = N_0 e^{-\frac{\ln 2}{139} t}$$

$$\Rightarrow \frac{N}{N_0} = e^{-\frac{\ln 2}{139} t}$$

$$\Rightarrow \ln\left(\frac{N}{N_0}\right) = -\frac{\ln 2}{139} t$$

$$\Rightarrow t = -\frac{139}{\ln 2} \ln\left(\frac{N}{N_0}\right)$$

95 % of the polonium is gone where $N = 0.05 N_0$. Hence

$$t_{95\%} = -\frac{139}{\ln 2} \ln(0.05) \approx 600 \text{ days}$$

Heat Transfer: Newton's Law of Cooling

Consider an object that is immersed in a medium of temperature T_s. Assume that medium can absorb any amount of heat without changing its temperature. Then a model for the rate of change of the objects' temperature T is

$$\frac{dT}{dt} = -k(T - T_s) \qquad (k > 0)$$

Assume $T(0) = T_0$, then

$$\frac{dT}{dt} = -k(T - T_s)$$

$$\Rightarrow \frac{dT}{T - T_s} = -k\, dt$$

$$\Rightarrow \int \frac{dT}{T-T_s} = -k \int dt$$

$$\Rightarrow \ln|T-T_s| = -kt + c$$

Assuming $T > T_s$, then $|T-T_s| = T-T_s$ and hence

$$T(t) = T_s + e^{-kt+c}$$

$$= T_s + e^c e^{-kt}$$

$$= T_s + \alpha e^{-kt} \qquad \alpha = e^c = \text{constant}$$

Since $T(0) = T_0$ then

$$T_0 = T_s + \alpha$$

$$\Rightarrow \alpha = T_0 - T_s$$

and thus

$$T(t) = T_s + (T_0 - T_s)e^{-kt}$$

This is called <u>Newton's law of cooling</u>

<u>The same law holds if $T < T_s$ except that now T will increase rather than decrease.</u>

Examples of where Newton's law of cooling holds is the temperature of your house if it is a temperature T_s outside and your heating goes off (I am assuming $T > T_s$ here). The cooling of a cup of coffee is another example. In both cases the assumption is that the outside air doesn't change temperature T_s as the house or coffee cools because there is a huge volume of air and it is constantly being convectively mixed so heat is quickly radiated away from the house or coffee.

Example

(#22, sec 6.5) An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65 °F. After 10 minutes, the beam warmed to 35° F and after another 10 minutes it was 50 °F. Use Newton's law of cooling to estimate the beam's initial temperature.

Solution:

We have:
$$T_s = 65\ ^\circ F$$
$$T(10\text{min}) = 35^\circ F$$
$$T(20\text{min}) = 50^\circ F$$

Assuming Newton's law of cooling:

$$T(t) = T_s + (T_o - T_s)e^{-kt}$$

$$\Rightarrow T(t) = 65 + (T_o - 65)e^{-kt}$$

$$\Rightarrow T_o - 65 = [T(t) - 65]e^{kt}$$

$$\Rightarrow T_o = 65 + [T(t) - 65]e^{kt}$$

We can get T_o from knowing $T(10) = 35$. However we also need to know k. To get it, note that

$$T_o = 65 + [35 - 65]e^{k10} \qquad \text{from } T(10) = 35$$
$$T_o = 65 + [50 - 65]e^{k20} \qquad \text{from } T(20) = 50$$

Equating the two equations (the left-hand sides of both are the same):

$$\cancel{65} + (35 - 65)e^{10k} = \cancel{65} + (50 - 65)e^{20k}$$

$$\Rightarrow -30e^{10k} = -15e^{20k}$$

$$\Rightarrow e^{10k} = \tfrac{1}{2}e^{20k}$$

$$\Rightarrow \ln e^{10k} = \ln\left(\tfrac{1}{2}e^{20k}\right)$$

$$\Rightarrow 10k = \ln\tfrac{1}{2} + 20k$$

$$\Rightarrow 10k = -\ln\tfrac{1}{2} = \ln 2$$

$$\Rightarrow k = \frac{\ln 2}{10}$$

and thus (again using $T(10) = 35$):

$$T_0 = 65 + (35-65)e^{\frac{\ln 2}{10}\cdot 10}$$

$$= 65 - 30e^{\ln 2}$$

$$= 65 - 30(2)$$

$$= 5\,^\circ F$$