APPM 1350: Section 6.6: L'Hôpital's Rule

When we first encountered limits we discussed the special case of

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

The problem here is that the limit of both the numerator and denominator as $x \to 0$ are zero, hence we have $\frac{0}{0}$ which is an example of an indeterminate form (other indeterminate forms are limits leading to $\frac{\infty}{\infty}$, $0 \cdot \infty$, or $\infty - \infty$). Yet despite

$$\frac{\lim_{x \to 0} \sin x}{\lim_{x \to 0} x} = \frac{0}{0}$$

the limit of the quotient is equal to one. We proved this using the sandwich theorem. We now introduce a very powerful technique that allows us to determine the limit of certain indeterminate forms quite directly.

L' Hôpital's Rule

If $f(x)$ and $g(x)$ either both approach 0 or both approach $\pm\infty$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

where "lim" stands for any of

$$\lim_{x \to \pm\infty} \qquad \lim_{x \to a} \qquad \lim_{x \to a^-} \qquad \lim_{x \to a^+}$$

Note that in the case of the limit at some finite point $x=a$ we require that $g'(x) \neq 0$ on an open interval around $x=a$ but excluding the point $x=a$ itself.

Example

Show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof:

Let $f(x) = \sin x$ and $g(x) = x$. Then $f'(x) = \cos x$ and $g'(x) = 1$. From L' Hôpital's rule:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

The proof of a weak form of L' Hôpital's rule is trivial. Specifically:

Thm: Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Proof:

Work backwards:

$$\frac{f'(a)}{g'(a)} = \frac{\left(\lim_{x \to a} \frac{f(x)-f(a)}{x-a}\right)}{\left(\lim_{x \to a} \frac{g(x)-g(a)}{x-a}\right)}$$

$$= \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x \to 0} \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$= \lim_{x \to 0} \frac{f(x)}{g(x)}$$

since $f(a) = g(a) = 0$.

∎

Proofs of the more complicated and stronger version of L'Hôpital's rule given on page 1 of my notes can be found in appendix 5 of Thomas.

Example

$$\lim_{x \to \infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$$

So using L'Hôpital's rule:

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{(1/x)}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$$

Example

$\frac{0}{0}$

$$\lim_{x \to 4} \frac{x^4 - 256}{x - 4} = \lim_{x \to 4} \frac{4x^3}{1}$$

$$= 4(4)^3$$

$$= 256$$

Example

$$\lim_{x\to 0} \frac{\cos 2x - \cos x}{\sin^2 x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-2\sin 2x + \sin x}{2\sin x \cos x}$$

$\frac{0}{0}$ again!

$$= \lim_{x\to 0} \frac{-4\cos 2x + \cos x}{2\cos^2 x - 2\sin^2 x}$$

$$= \frac{-4\cdot 1 + 1}{2 - 2\cdot 0}$$

$$= -\frac{3}{2}$$

So, if you apply L'Hôpital's rule and you get another indefinite form, just keep applying it. Just remember, L'Hôpital's rule <u>only</u> works for limits that produce $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$.

Example

$$\lim_{x\to 0} \frac{\ln \sec 2x}{\ln \sec x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\frac{1}{\cancel{\sec 2x}}\, 2\cancel{\sec 2x}\tan 2x}{\frac{1}{\cancel{\sec x}}\, \cancel{\sec x}\tan x}$$

$$= 2\lim_{x\to 0} \frac{\tan 2x}{\tan x}$$

$\frac{0}{0}$ again

$$= 2\lim_{x\to 0} \frac{2\sec^2 2x}{\sec x}$$

$$= 4$$

Example

$$\frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{x^4+x^2}{e^x+1} = \lim_{x\to\infty} \frac{4x^3+2x}{e^x}$$

$$\frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{12x^2+2}{e^x}$$

$$\frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{24x}{e^x}$$

$$\frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{24}{e^x}$$

$$= 0$$

Limits Involving Indeterminate Products and Differences

L'Hôpital's rule only applies to cases where the indeterminate form is $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$. However, other indeterminate forms arise, specifically $0\cdot\infty$ and $\infty-\infty$. To compute limits involving these indeterminate forms we convert the limit into a form for which L'Hôpital's rule does apply.

Example

(# 32, sec 6.6)

$$-\infty-(-\infty) = -\infty+\infty = \infty-\infty$$

$$\lim_{x\to 0^+} (\ln x - \ln \sin x) = \lim_{x\to 0^+} \ln\left(\frac{x}{\sin x}\right)$$

$$\frac{0}{0}$$

$$= \ln\left[\lim_{x\to 0^+} \frac{x}{\sin x}\right]$$

$$= \ln\left[\lim_{x\to 0^+} \frac{1}{\cos x}\right]$$

$$= \ln 1$$

$$= 0$$

Example

$0\cdot\infty$ $\qquad$ $\frac{0}{0}$

$$\lim_{x\to 0} x\csc x = \lim_{x\to 0} \frac{x}{\sin x}$$

$$= \lim_{x\to 0} \frac{1}{\cos x}$$

$$= 1$$

Example

$-\infty\cdot 0$ $\qquad$ $-\frac{\infty}{\infty}$

$$\lim_{x\to -\infty} x^2 e^x = \lim_{x\to -\infty} \frac{x^2}{e^{-x}}$$

$-\frac{\infty}{\infty}$

$$= \lim_{x\to -\infty} \frac{2x}{-e^{-x}}$$

$$= 2\lim_{x\to -\infty} \frac{2}{e^{-x}}$$

$$= 2\lim_{x\to -\infty} 2e^x$$

$$= 0$$

Limits Involving Indeterminate Powers

Another example of an indeterminate form are limits leading to 0^0, ∞^0, and 1^∞. In these cases then we take the log and apply L'Hôpital's rule to it. Specifically we use the fact that

> If $\lim_{x\to a} \ln[f(x)] = L$ then
>
> $$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L$$
>
> where a is either finite or infinite.

Example

Find $\lim\limits_{x \to 0^+} (1+x)^{1/x}$

Solution:

This limit leads to the indefinite form 1^{∞}. So instead try:

$$y(x) = (1+x)^{1/x}$$

$$\Rightarrow \ln y(x) = \frac{1}{x} \ln(1+x)$$

$\infty \cdot 0$

$$\Rightarrow \lim_{x \to 0^+} \left[\ln y(x)\right] = \lim_{x \to 0^+} \frac{1}{x} \ln(1+x)$$

$$= \lim_{x \to 0^+} \frac{\ln(1+x)}{x} \quad \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{\left(\frac{1}{1+x}\right)}{1}$$

$$= 1$$

$$\Rightarrow \operatorname{limit}_{x \to 0^+} y(x) = \lim_{x \to 0^+} (1+x)^{1/x} = e^1 = e$$

So

$$\lim_{x \to 0^+} (1+x)^{1/x} = e$$

which is a somewhat remarkable result.

Example

Find $\lim_{x \to 0^+} x^x$

Solution:

We have 0^0 now. So, let

$$y = x^x$$

$$\Rightarrow \ln y = x \ln x$$

which is a $0 \cdot \infty$ indeterminate form (this is always the case with 0^0, ∞^0, and 1^∞ indeterminate forms). So:

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x \ln x$$

$$= \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} \quad \leftarrow -\frac{\infty}{\infty}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)}$$

$$= -\lim_{x \to 0^+} x$$

$$= 0$$

Hence $\lim_{x \to 0^+} y = \lim_{x \to 0^+} x^x = e^0 = 1$

Example

Find $\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x}$

Solution:

This is a ∞^0 indeterminate form. So:

$$y = (\tan x)^{\cos x}$$

$$\Rightarrow \ln y = \cos x \ln(\tan x)$$

Hence:

0 · ∞ as expected, so rewrite

$$\lim_{x \to \frac{\pi}{2}^-} \ln y = \lim_{x \to \frac{\pi}{2}^-} \cos x \cdot \ln(\tan x)$$

$\frac{\infty}{\infty}$ now

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\ln(\tan x)}{\sec x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{1}{\tan x} \sec^{\cancel{2}} x}{\cancel{\sec x} \tan x}$$

$\frac{\infty}{\infty}$ again

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\sec x}{\tan^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\sec x \cancel{\tan x}}{2 \cancel{\tan x} \sec^2 x}$$

$$= \frac{1}{2} \lim_{x \to \frac{\pi}{2}^-} \cos x$$

$$= 0$$

Thus: $\lim_{x \to \frac{\pi}{2}^-} y = \lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x} = e^0 = 1.$