APPM 1350: Section 6.7: Relative Rates of Growth

In Monday's lecture in class we discussed the notion of one function growing faster than or at the same rate as another as we take the limit. In this section we explore this idea in more detail. In particular we wish to discuss functions that are positive as x becomes large. So:

Def Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1) f grows faster than g as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$$

or equivalently, if

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0$$

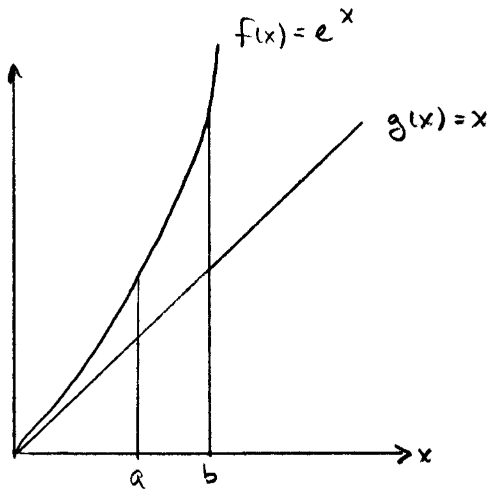
We also say that $g(x)$ grows slower than f as $x \to \infty$.

2) f and g grow at the same rate as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$$

where $L \neq 0$ and $|L| < \infty$.

This "definition" encodes an intuitively obvious idea. For example, consider $f(x) = e^x$ and $g(x) = x$. Graph these for positive x.



It is immediately obvious that $f(x) = e^x$ is getting larger (i.e. growing) faster than $g(x) = x$ as x increases beause e^x is concave up while x is not. Hence the ratio

$$\frac{f(x)}{g(x)}$$

is clearly smaller that the same ratio a $x = b$, i.e.,

$$\frac{f(b)}{g(b)} > \frac{f(a)}{g(a)}$$

This ratio just keeps getting larger and larger, so

$$\lim_{x \to \infty} \frac{e^x}{x} = \infty$$

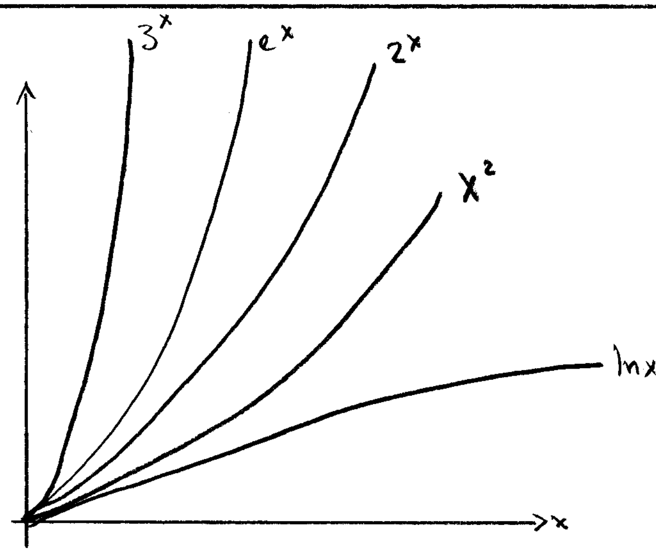
On the otherhand, something like $f(x) = 2x$ and $g(x) = x$ have the same convexity and so asymptotically they grow at the same rate. Put differently, the relative value of $f(x)$ with respect to $g(x)$ is a constant, which means

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{2x}{x} = 2$$

The limit is a finite, but nonzero, number.

As a practical matter:

1) a^x grows faster than b^x if $a>b$
2) a^x grows faster than x^n if $a>1, x>0, n>0$
3) x^n grows faster than $\ln x$ for $x>0$ and $n>0$



To prove this;

1) $$\lim_{x\to\infty} \frac{a^x}{b^x} = \lim_{x\to\infty} \left(\frac{a}{b}\right)^x = \infty \qquad \text{if } a>b$$

2) For simplicity assume n is a positive integer. Then

$$\lim_{x\to\infty} \frac{a^x}{x^n} = \lim_{x\to\infty} \frac{e^{x\ln a}}{x^n} \qquad \left(\tfrac{\infty}{\infty}\text{, use L'Hôpital's rule}\right)$$

$$= \lim_{x\to\infty} \frac{\ln a\, e^{x\ln a}}{n x^{n-1}} \qquad \left(\tfrac{\infty}{\infty}\text{ again, keep using L'Hôspital's rule}\right)$$

$$\vdots$$

$$= \lim_{x\to\infty} \frac{(\ln a)^n e^{x\ln a}}{n!} = \infty \qquad (n! = n(n-1)(n-2)\cdots 1)$$

The proof for $n > 0$ but not an integer is similar.

3) $$\lim_{x \to \infty} \frac{x^n}{\ln x} = \lim_{x \to \infty} \frac{n x^{n-1}}{\left(\frac{1}{x}\right)} \qquad \text{(L'Hôpital's rule)}$$

$$= \lim_{x \to \infty} n x^n$$

$$= \infty$$

The definition on page 1 also allows one to look at the relative growth rates of any positive function as $x \to \infty$.

<u>Example</u>

(#2a, see 6.7)

$$\lim_{x \to \infty} \frac{10x^4 + 30x + 1}{e^x} = \lim_{x \to \infty} \frac{40x^3 + 30}{e^x}$$

$$= \lim_{x \to \infty} \frac{120x^2}{e^x}$$

$$= \lim_{x \to \infty} \frac{240x}{e^x}$$

$$= \lim_{x \to \infty} \frac{240}{e^x}$$

$$= 0$$

So $10x^4 + 30x + 1$ grows more slowly than e^x as $x \to \infty$. As a general rule, positive polynomials always grow slower than e^x as $x \to \infty$ (or more generally as a^x for $a > 1$).

Example

(#2h, see 6.7)

$$\lim_{x\to\infty} \frac{e^{x-1}}{e^x} = \lim_{x\to\infty} \frac{e^x e^{-1}}{e^x}$$

$$= \frac{1}{e} \lim_{x\to\infty} \frac{e^x}{e^x}$$

$$= \frac{1}{e}$$

Thus e^x and e^{x-1} grow at the same relative rate as $x \to \infty$.

Example

$$\underset{x\to\infty}{\text{limit}} \frac{2x^2+3x+4}{\ln x} = \lim_{x\to\infty} \frac{4x+3}{\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} 4x^2+3x$$

$$= \infty$$

So $2x^2+3x+4$ grows faster than $\ln x$ as $x \to \infty$. As a general rule all positive polynomials grow faster than $\ln x$ as $x \to \infty$.

Order and Oh-Notation

Def A function f(x) is <u>of smaller order</u> than g as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

We indicate that by writing $f = o(g)$, read "f is of order little-oh of g".

So $\ln x = o(x^n)$ for any $n > 0$ because

$$\operatorname*{limit}_{x \to \infty} \frac{\ln x}{x^n} = \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{nx^{n-1}}$$

$$= \lim_{x \to \infty} \frac{1}{nx^n}$$

$$= 0$$

Def Let f(x) and g(x) be positive for x sufficiently large. Then f is <u>of at most the order of g</u> as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \leq M$$

for all x greater than some x_0. We indicate this by writing $f = O(g)$, read "f is of order big-oh of g".

So $2x = O(x)$ because

$$\frac{2x}{x} \le 2 \qquad \text{(actually equal to 2 in this case)}$$

for all $x > 0$.

A more interesting case is (10b, see 6.2)

$$\frac{1}{x} + \frac{1}{x^2} = O\left(\frac{1}{x}\right)$$

Proof:

$$f(x) = \frac{1}{x} + \frac{1}{x^2}$$

$$g(x) = \frac{1}{x}$$

Then $$\frac{f(x)}{g(x)} = \frac{\frac{1}{x} + \frac{1}{x^2}}{\frac{1}{x}} = 1 + \frac{1}{x} \le 2 \quad \text{for all } x > 1$$

When we say that $f = o(g)$, the implication is that $f(x)$ grows slower than $g(x)$, but we don't make any claim about how much slower. $f = O(g)$ is actually implied by $f = o(g)$, but usually in practice it is used to suggest something about how much slower f grows than g.

These definitions are made more precise when you learn about power series expansions in calculus 2. For now just treat them as definitions so you will know what somebody means when they say something like the error in an approximation is $O(x^2)$. By that they mean the error grows no faster than x^2 grows.