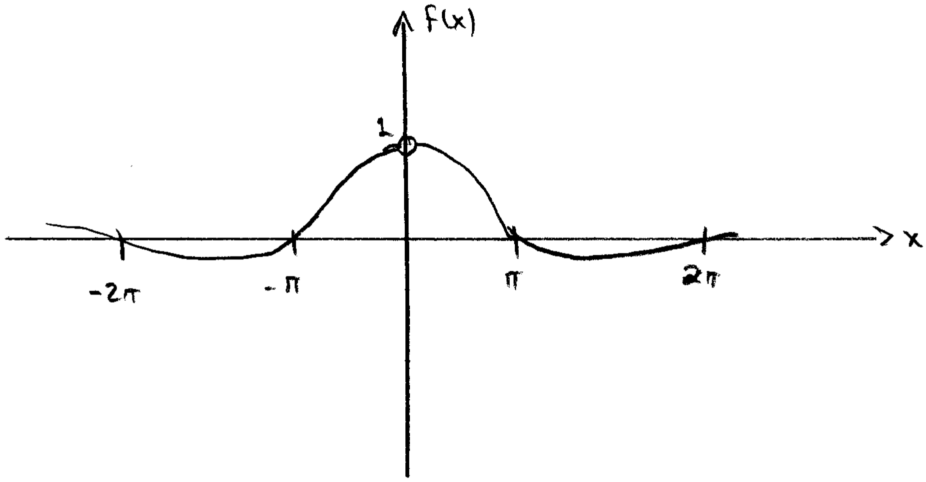
# APPM 1350: Sections 1.1 & 1.2: Rates of Change and Limits; Finding Limits

## Limits

Consider the function

$$f(x) = \frac{\sin x}{x} \qquad x \neq 0$$

This is a well behaved function except at $x=0$ where it is undefined. If you plot this function it looks like:




The interesting thing is that while $f(x)$ is not defined at $x=0$, as we get closer and closer to $x=0$ the value of the function gets closer and closer to 1. Mathematically we say $\frac{\sin}{x}$ approaches a <u>limit</u> of 1 as $x$ approaches 0. We denote this by writing

$$\underset{x \to 0}{\text{limit}} \frac{\sin x}{x} = 1$$

or

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

or sometimes

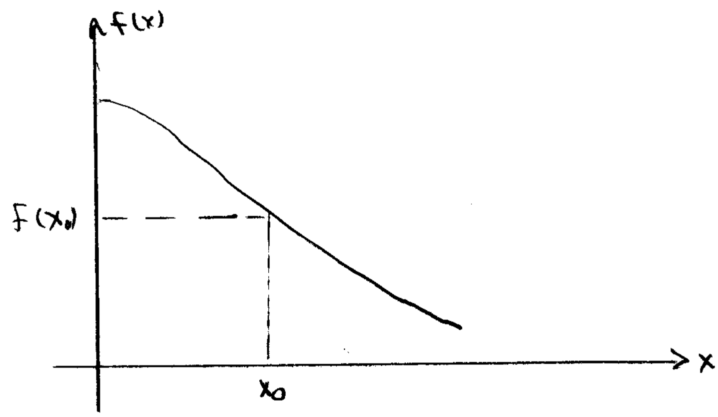
$$\frac{\sin x}{x} \to 1 \quad \text{as} \quad x \to 0$$

We can <u>informally</u> define the concept of a limit with the following definition.

| |
|---|
| Def: Let $f(x)$ be defined on an open interval about $x_0$, except possibly at $x_0$ itself. If $f(x)$ gets arbitrarily close to a value $L$ for as $x$ gets closer and closer to $x_0$, we say that $f(x)$ approaches the limit $L$ as $x$ approaches $x_0$, and we write $$\lim_{x \to x_0} f(x) = L$$ |

If a function has an unbroken graph (i.e., it is continuous) around a point $x_0$, the the limit as $x \to x_0$ is just $f(x_0)$:

$$\lim_{x \to x_0} f(x) = f(x_0)$$




For the moment we will limit our attention to case such as this or the example on the previous page where the functions are unbroken (i.e., continuous) except perhaps at a single point. In section 1.4 next week we will extend our discussion of limits to more pathological functions.

# Evaluating Limits

So let's explore by example some ways to evaluate limits.

## Example

Find $\lim_{x \to 3} x^2$ if it exists.

Soln: $x^2$ is defined at $x=3$, so the limit is just the function defined at that point, i.e.,

$$\lim_{x \to 3} x^2 = 9$$

## Example

Find $\lim_{x \to 0} e^x$ if it exists.

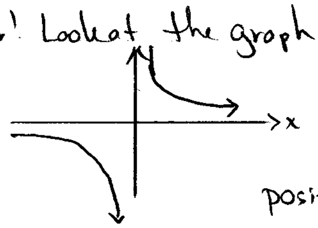
Soln: This is similiar to the example above, the limit "$e^x$" evaluated at zero:

$$\lim_{x \to 0} e^x = 1$$

## Example

Find $\lim_{x \to 0} \frac{1}{x}$ if it exists.

Soln: Here we have a case where the function is undefined at $x=0$ where we want to evaluate the limit. But does the function approach a constant value as $x$ gets closer and closer to zero? No! Look at the graph of the function. As $x \to 0$ from the left




$f(x)$ becomes ever more negative and as $x \to 0$ from the right it becomes ever more positive. So the $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Example

Find $\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4}$ if it exists.

Soln: Here we have an interesting situation. The denominator is zero at $x = 4$. But so is the numerator. Strictly speaking this function is undefined at $x = 4$ because you can never divide by zero. But in this case appearances are deceiving. Factor the numerator and you find you can get rid of the denominator. So

$$\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4} = \lim_{x \to 4} \frac{\cancel{(x-4)}(x+3)}{\cancel{(x-4)}}$$

$$= \lim_{x \to 4} (x+3)$$

$$= 7$$

Example

Find $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$ if it exists.

Soln: Same game as above:

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x^2 + x + 1)$$

$$= 1 + 1 + 1$$

$$= 3$$

# Properties of Limits

Thm: Let $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$. Then

1) $\lim\limits_{x \to c} [f(x) + g(x)] = L + M$

2) $\lim\limits_{x \to c} [f(x) - g(x)] = L - M$

3) $\lim\limits_{x \to c} f(x)\, g(x) = LM$

4) $\lim\limits_{x \to c} k f(x) = k \lim\limits_{x \to c} f(x) = kL \qquad (k \in \mathbb{R})$

5) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ provided $M \neq 0$

6) If $m, n \in \mathbb{Z}$, then $\lim\limits_{x \to c} [f(x)]^{m/n} = L^{m/n}$

provided $L^{m/n} \in \mathbb{R}$

## Example

$$\lim_{x \to 4} \frac{x+4}{3x} = \frac{\lim\limits_{x \to 4} (x+4)}{\lim\limits_{x \to 4} 3x}$$

$$= \frac{8}{12}$$

$$= \frac{2}{3}$$

Example

$$\lim_{x \to 2} x^2 = \left(\lim_{x \to 2} x\right)^2$$

$$= 2^2$$

$$= 4$$

## The Sandwich Theorem

Let's return to our original example of $f(x) = \frac{\sin x}{x}$. Here there are no tricks for getting rid of the zero in the denominator as $x \to 0$. So how do we evaluate

$$\lim_{x \to 0} \frac{\sin x}{x}$$

or more specifically prove that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

which is what we asserted on the first page. Well, consider the following. First note the following inequality:

$$\cos x < \frac{\sin x}{x} < 1 \qquad |x| \in \left(0, \frac{\pi}{2}\right) \qquad (1)$$

and also

$$0 < \left|\frac{\sin x}{x}\right| < 1 \qquad (2)$$

$$\Rightarrow |\sin x| < |x| \qquad x \neq 0$$

So using (1) and (2) and restricting $|x| \in (0, \frac{\pi}{2})$:

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x = 2\sin^2\frac{x}{2} \leq 2\left|\sin\frac{x}{2}\right| \leq 2\,\frac{|x|}{2} = |x|$$

i.e., we have established

$$0 < 1 - \frac{\sin x}{x} < |x|$$

or more generally

$$0 \leq 1 - \frac{\sin x}{x} \leq |x|$$

$$\Rightarrow \lim_{x \to 0} 0 \leq \lim_{x \to 0}\left(1 - \frac{\sin x}{x}\right) \leq \lim_{x \to 0} |x|$$

$$\Rightarrow 0 \leq \lim_{x \to 0}\left(1 - \frac{\sin x}{x}\right) \leq 0$$

We have bracketed $\lim_{x \to 0} 1 - \frac{\sin x}{x}$ from the top and bottom by zero, so clearly:

$$\lim_{x \to 0}\left(1 - \frac{\sin x}{x}\right) = 0$$

$$\Rightarrow \lim_{x \to 0} 1 - \lim_{x \to 0} \frac{\sin x}{x} = 0$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$\blacksquare$

This is an example of the <u>sandwich theorem</u>

> **Thm:** Suppose that $g(x) \le f(x) \le h(x)$ $\forall$ $x$ in some open interval containing $c$, except possibly at $x=c$ itself. Suppose also that
>
> $$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$
>
> Then $\lim_{x \to c} f(x) = L$

Finally, let's explore one other technique for finding limits. First start with the following example...

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \qquad (x \in \mathbb{R})$$

We have another division by zero problem. But:

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

Cool. But how about this case?

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \qquad (x > 0)$$

You can't expand the radical. But you can change the numerator into a polynomial by a clever trick and then reduce it as in the previous example:

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

# Average Rates of Change and Secant Lines

Consider the example we just finished. We were working with the function

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Let

$$x_1 = x$$
$$x_2 = x+h$$
$$f(x) = \sqrt{x}$$

then we can rewrite our function as

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$= \frac{\Delta f}{\Delta x}$$

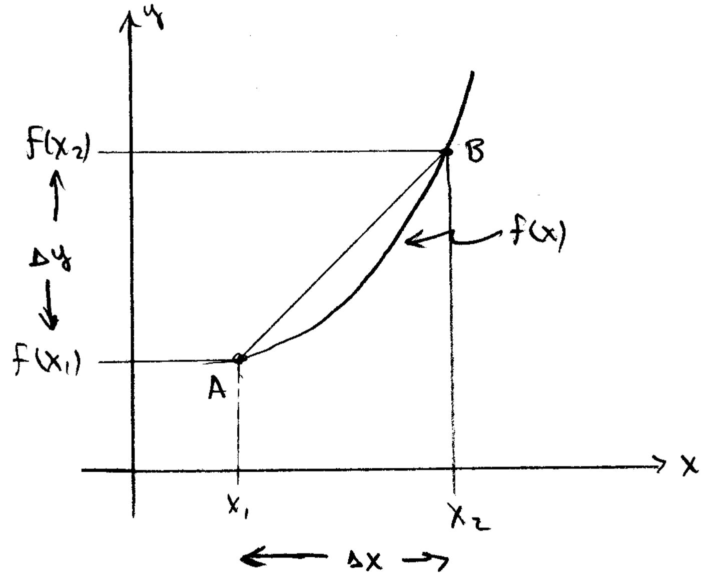
This is the average rate of change of $f(x)=\sqrt{x}$ in going from the point $x_1$ to the point $x_2$.

> Def: The average rate of change of $y=f(x)$ with respect to $x$ over an interval $[x_1, x_2]$ is:
>
> $$\frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

An example of this in physics is speed, defined as the average rate of change of some distance $\Delta x$ covered in some time increment $\Delta t$:

$$\text{speed} = \frac{\Delta x}{\Delta t}$$

Now consider some arbitrary graph




If we draw a straight line between the point A at $(x_1, f(x_1))$ and the point B at $(x_2, f(x_2))$, its slope is just

$$\text{slope} = \frac{\Delta y}{\Delta x}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This straight line is called a <u>secant line</u> and its slope is by definition the average rate of change of the function $f(x)$ in going from $x_1$ to $x_2$.

What is interesting about secant lines is that as we let the point $x_2$ get closer and closer to $x_1$ (for appropriate functions) the slope often approaches a constant value. That is, the slope has a limit as $x_2 \to x_1$. This is in fact what we proved on page 9 for the function $\sqrt{x}$:

$$\lim_{x_2 \to x} \frac{\sqrt{x_2} - \sqrt{x}}{x_2 - x} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \frac{1}{2\sqrt{x}} \qquad (x > 0)$$

This limit is called the <u>derivative</u> of f(x) at x and is the <u>instantaneous rate of change</u> of f(x) at the point x. To see graphically that the slope of the secant line approaches a limit for the arbitrary function on the previous page, just draw in different secant lines.