APPM 1350: Section 1.3: Target Values and Formal Definition of Limits

Class exercize

(P. 42, page 66). Find

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

for $f(x)=\sqrt{3x+1}$ at $x=0$.

Solution:

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{3h+1}-1}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{3h+1}-1)(\sqrt{3h+1}+1)}{h(\sqrt{3h+1}+1)}$$

$$= \lim_{h\to 0} \frac{(3h+1) \quad -1}{h(\sqrt{3h+1}+1)}$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}(\sqrt{3h+1}+1)}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3h+1}+1}$$

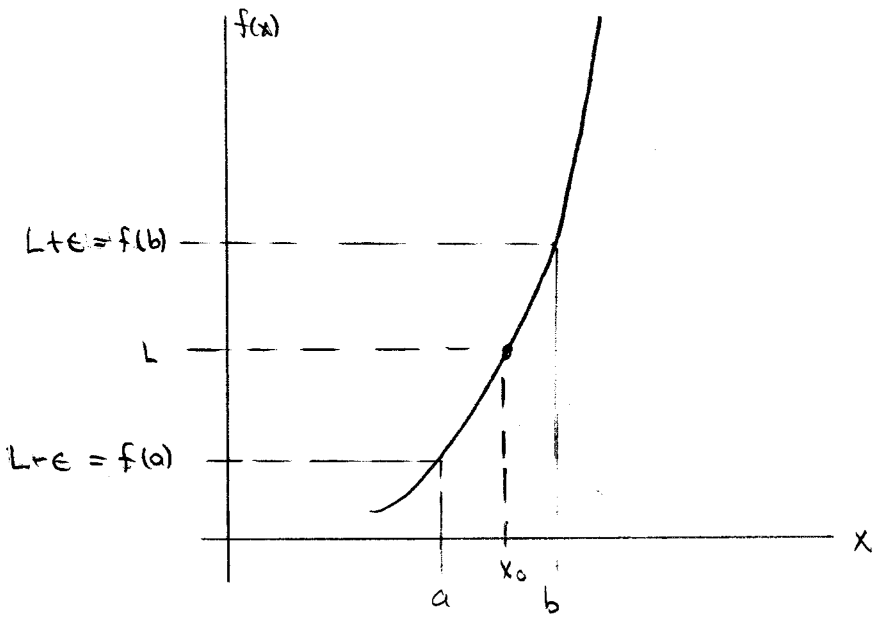
$$= \frac{3}{2}$$

The Precise Definition of a Limit

Recall our informal definition of a limit:

> Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0 we say that f approaches the limit L as x approaches x_0, and we write
>
> $$\lim_{x \to x_0} f(x) = L$$

So what do we mean by arbitrarily or sufficiently close? Well, consider the following graph:



We have a smooth function $f(x)$, i.e., no breaks except for perhaps at x_0. And if we were to fill in a break at $f(x_0)$ with the value L then the function truly wouldn't have any breaks. Now let $\epsilon > 0$ be some arbitrary but generally very small number. This is a tolerance in how far we want $f(x)$ to get from L. In particular we want to consider some set of $f(x)$ values so that

$$|f(x) - L| < \epsilon$$

Now corresponding to $f(x) = L - \epsilon$ will be some $x = a$ and corresponding to $f(x) = L + \epsilon$ will be some $x = b$. Clearly from our graph, if

$$a < x < b$$

then indeed

$$|f(x) - L| < \epsilon$$

Note that $x \in (a,b)$. Also note that while $f(a)$ and $f(b)$ are equidistant from L, a and b may not themselves be equidistant from x_0. But since $|f(x) - L| < \epsilon$ for all $x \in (a,b)$, then if we choose some smaller interval $(x_0 - \delta, x_0 + \delta)$ where

$$(x_0 - \delta, x_0 + \delta) \subseteq (a, b)$$

then $x \in (x_0 - \delta, x_0 + \delta)$ still means $|f(x) - L| < \epsilon$. We have simply restricted the allowable x's such that $f(x)$ is within ϵ of L. We can restate this as saying that for all x such that $|x - x_0| < \delta$ then $|f(x) - L| < \epsilon$. Note that

$$\delta = \min(|a - x_0|, |b - x_0|)$$

So for these smooth functions, if we choose any $\epsilon > 0$, no matter how small, there will be some corresponding δ such that so long as $x \in (x_0 - \delta, x_0 + \delta)$ then $f(x)$ will be within a distance ϵ of L. This is the formal definition of a limit.

Def: Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0 and write

$$\lim_{x \to x_0} f(x) = L$$

if for every $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x where

$$|x - x_0| < \delta$$

then

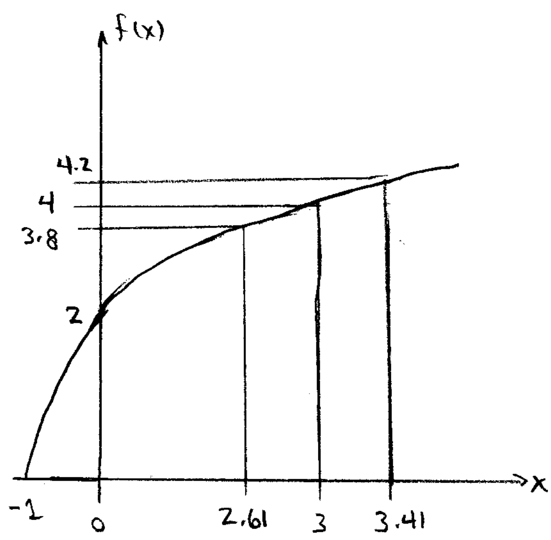
$$|f(x) - L| < \epsilon$$

Let's now explore this concept with some examples.

Example

(#10, page 74)

Consider $f(x) = 2\sqrt{x+1}$



Choose $x_0 = 3$

$L = 4$

$\epsilon = 0.2$

Find a $\delta > 0$ such that $0 < |x-3| < \delta \Rightarrow |f(x) - 4| < \epsilon = 0.2$

Solution:

From the graph

$$a = 2.61 \Rightarrow f(a) = 3.8$$

and

$$b = 3.41 \Rightarrow f(b) = 4.2$$

Also: $|f(a) - 4| = |f(b) - 4| = \epsilon = 0.2$. Clearly for <u>any</u> x such that

$$2.61 < x < 3.41$$

then

$$|f(x) - 4| < 0.2$$

Let's tighten this interval on x so that our interval is symmetric around $x_0 = 3$. Choose

$$\delta = \min(|x_0 - a|, |x_0 - b|)$$

$$= \min(|3 - 2.61|, |3 - 3.41|)$$

$$= \min(0.39, 0.41)$$

$$= 0.39$$

So for <u>all</u> x such that

$$|x - 3| < 0.39 \quad \text{then} \quad |2\sqrt{x+1} - 4| < 0.2$$

Example

(#18, page 75)

Suppose we have $f(x) = \sqrt{x}$, $L = \frac{1}{2}$, $x_0 = \frac{1}{4}$, and $\epsilon = 0.1$. Find a corresponding $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$$

Solution:

We do this one algebraically. From

$$|f(x) - L| < \epsilon$$

we have

$$\left|\sqrt{x} - \frac{1}{2}\right| < 0.1$$

$$\Rightarrow -0.1 < \sqrt{x} - \frac{1}{2} < 0.1$$

$$\Rightarrow \frac{1}{2} - 0.1 < \sqrt{x} < \frac{1}{2} + 0.1$$

$$\Rightarrow 0.4 < \sqrt{x} < 0.6$$

$$\Rightarrow 0.16 < x < 0.36$$

Now choose

$$\delta = \min\left(|0.16 - x_0|, |0.36 - x_0|\right)$$

$$= \min\left(|0.16 - 0.25|, |0.36 - 0.25|\right)$$

$$= \min(0.09, 0.11)$$

$$= 0.09$$

Hence:

$$|x - \tfrac{1}{4}| < 0.09 \implies |\sqrt{x} - \tfrac{1}{2}| < 0.1$$

Proving Limits

Example

Prove $\lim_{x \to \frac{1}{4}} \sqrt{x} = \frac{1}{2}$

Proof:

This is essentially the same as the previous example except there we found δ for a specific case of $\epsilon = 0.2$. Now we let ϵ be any value $\epsilon > 0$, So; we start with

$$|\sqrt{x} - \tfrac{1}{2}| < \epsilon$$

$$\implies -\epsilon < \sqrt{x} - \tfrac{1}{2} < \epsilon$$

$$\implies \tfrac{1}{2} - \epsilon < \sqrt{x} < \tfrac{1}{2} + \epsilon$$

$$\implies \left(\tfrac{1}{2} - \epsilon\right)^2 < x < \left(\tfrac{1}{2} + \epsilon\right)^2$$

Choose

$$\delta = \min\left(\left|\left(\tfrac{1}{2} - \epsilon\right)^2 - \tfrac{1}{4}\right|, \left|\left(\tfrac{1}{2} + \epsilon\right)^2 - \tfrac{1}{4}\right|\right.$$

$$= \min\left(|\epsilon^2 - \epsilon|, |\epsilon^2 + \epsilon|\right)$$

$$= \epsilon[\min\left(|\epsilon - 1|, |\epsilon + 1|\right)$$

$$= \epsilon|1 - \epsilon|$$

So:

Given <u>any</u> $\epsilon > 0$, then for <u>all</u> x such that

$$|x - \tfrac{1}{4}| < \epsilon |1 - \epsilon|$$

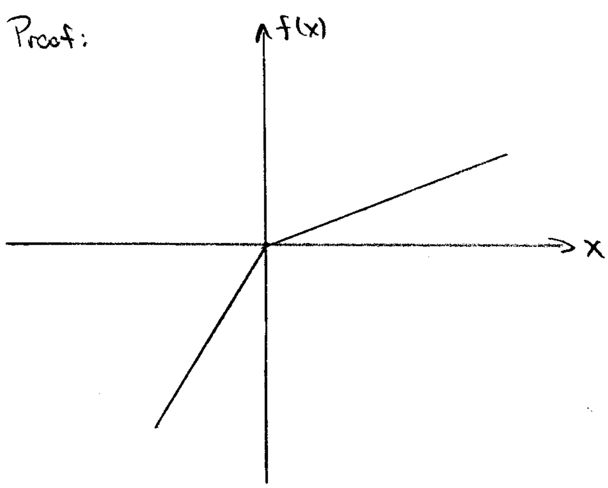
we have that

$$|\sqrt{x} - \tfrac{1}{2}| < \epsilon$$

<u>Example</u>

Prove that $\lim_{x \to 0} f(x) = 0$ if $f(x) = \begin{cases} 2x & x < 0 \\ \frac{x}{2} & x \geq 0 \end{cases}$

Proof:



Assume some arbitrary $\epsilon > 0$. Then consider

$$|f(x) - 0| < \epsilon$$

$$\Rightarrow |f(x)| < \epsilon$$

$$\Rightarrow -\epsilon < f(x) < \epsilon$$

Split this into two cases:

$$-\epsilon < f(x) < 0 \quad \text{and} \quad 0 \leq f(x) < \epsilon$$

For the first case:

$$-\epsilon < 2x < 0 \Rightarrow -\frac{\epsilon}{2} < x < 0$$

and for the second case

$$0 \leq \frac{x}{2} < \epsilon \Rightarrow 0 \leq x \leq 2\epsilon$$

Combining them;

$$-\frac{\epsilon}{2} < x < 2\epsilon$$

Then choose

$$\delta = \min\left(\left|0 - \left(-\frac{\epsilon}{2}\right)\right|, \left|0 - 2\epsilon\right|\right)$$

$$= \min\left(\frac{\epsilon}{2}, 2\epsilon\right)$$

$$= \frac{\epsilon}{2}$$

Hence, for <u>any</u> $\epsilon > 0$, then

for all $x \ni 0 < |x - 0| < \frac{\epsilon}{2} \Rightarrow |f(x) - 0| < \epsilon$

means "such that"

or rewriting:

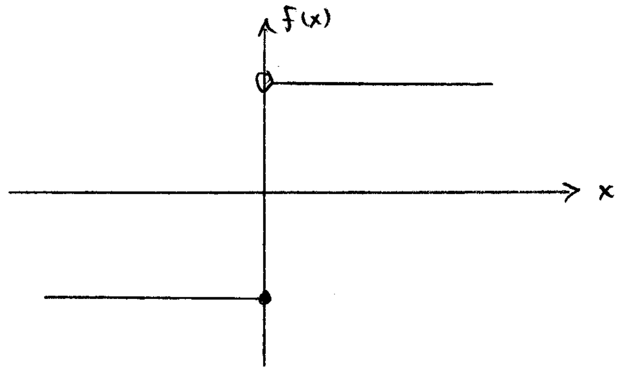
$$0 < |x| < \frac{\epsilon}{2} \Rightarrow |f(x)| < \epsilon$$

$\blacksquare$

An Example When a Limit Doesn't Exist

Where do these ϵ, δ ideas fall apart if a limit doesn't exist? Consider

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



There is no limit as $x \to 0$. But suppose we assumed a limit did exist at $x=0$ and its value was $L=0$. If we chose $\epsilon > 1$, then

$$|f(x) - 0| < 1$$

could be satisfied by literally <u>any</u> $\delta > 0$. That is, any $|x-0| < \delta > 0$ would be sufficient for $f(x)$ to be within a value 1 of 0. But for any $\epsilon < 1$ there is no $\delta > 0$ such that $|f(x) - 0| < \epsilon$ for <u>any</u> $|x - 0| < \delta$. This is simply because the break in the function at $x=0$ prevents us from finding a $f(x)$ such that it is less than one unit distance away from zero.

More Examples

Prove that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Proof:

Assume that $\sin\left(\frac{1}{x}\right)$ does have a limit equal to c at $x=0$. Then suppose we arbitrarily we to choose $\epsilon = \frac{1}{2}$. If the limit at $x=0$ exists and equals c then we ought to be able to find some δ such that $0 < |x-0| < \delta \Rightarrow \left|\sin\left(\frac{1}{x}\right) - c\right| < \epsilon = \frac{1}{2}$. So:

$$\left|\sin\left(\frac{1}{x}\right) - c\right| < \frac{1}{2}$$

Suppose we were to choose

$$x_1 = \frac{1}{(2n+\frac{1}{2})\pi}$$

and

$$x_2 = \frac{1}{(2n-\frac{1}{2})\pi}$$

where $n \in \mathbb{Z}$. Now, for any δ we can always find an n such that

$$0 < |x_1| < \delta \quad \text{and} \quad 0 < |x_2| < \delta$$

(the smaller the δ the larger $|n|$). Now note that

$$\sin\left(\frac{1}{x_1}\right) = \sin\left[(2n+\tfrac{1}{2})\pi\right] = \sin\frac{\pi}{2} = 1$$

$$\sin\left(\frac{1}{x_2}\right) = \sin\left[(2n-\tfrac{1}{2})\pi\right] = \sin\left(-\frac{\pi}{2}\right) = -1$$

Then if c really is a limit we would have

$$\left| \sin \frac{1}{x_1} - c \right| = |1 - c| < \epsilon = \frac{1}{2} \qquad (1)$$

and

$$\left| \sin \frac{1}{x_2} - c \right| = |1 + c| < \epsilon = \frac{1}{2} \qquad (2)$$

But note that

$$|1-c| + |1+c| \geq |1 - c + 1 + c| = 2 \qquad \text{(triangle inequality)}$$

and

$$|1-c| + |1+c| < \frac{1}{2} + \frac{1}{2} = 1 \qquad \text{(from 1 and 2 above)}$$

This would imply that

$$2 \leq |1-c| + |1+c| < 1$$

This is an absurdity. and hence there is no limit at $x = 0$.

$\blacksquare$

Example

(# 46, p. 75) Prove $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

Proof:

Choose $\epsilon > 0$ and assume $x \neq 1$. Then

$$\left| \frac{x^2-1}{x-1} - 2 \right| < \epsilon$$

$$\Rightarrow \left| \frac{(x+1)(x-1)}{(x-1)} - 2 \right| < \epsilon$$

$$\Rightarrow |x+1-2| < \epsilon$$

$$\Rightarrow |x-1| < \epsilon$$

$$\Rightarrow -\epsilon < x-1 < \epsilon$$

$$\Rightarrow 1-\epsilon < x < 1+\epsilon$$

Choose

$$\delta = \min(|1-\epsilon-1|, |1+\epsilon-1|)$$

$$= \min(|-\epsilon|, |\epsilon|)$$

$$= |\epsilon|$$

$$= \epsilon$$

Hence $\forall\, \epsilon > 0$

$$0 < |x-1| < \epsilon \Rightarrow \left| \frac{x^2-1}{x+1} - 2 \right| < \epsilon$$