APPM 1350: Section 1.5: Continuity

Today's lecture will discuss continuity, a fancy term for saying that the graph of a function has no breaks in it. But before we do, I want to digress and present some useful results on inequalities and limits.

Digression: A Couple Useful Inequalities

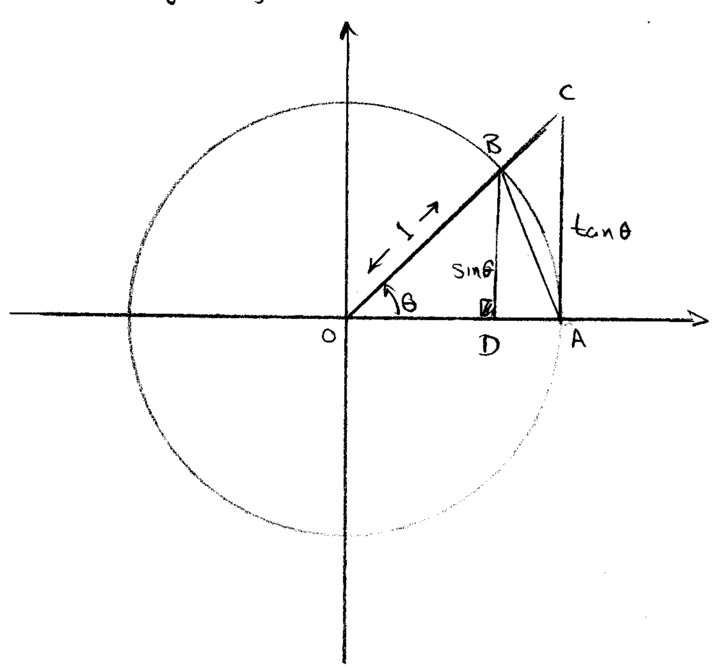
The following inequality can be <u>very</u> useful!

<u>Theorem</u>: For $0 < |\theta| < \pi/2$;

$$\cos\theta < \frac{\sin x}{x} < 1$$

Proof:

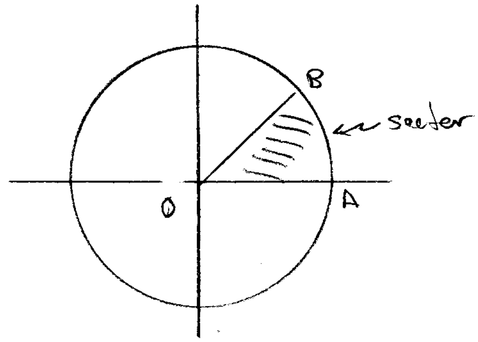
Consider the following diagram. The circle has a radius of one (i.e, unit radius)



Here we have $0 < \theta < \pi/2$. By inspection:

area of triangle OAB < area of sector OAB < area of triangle OAC

Note: the sector OAB is the piece of the circle subtended by the angle θ



The area of the circle is πr^2 where r is the radius. Since $r=1$, the area of the whole circle is π. As for the area of the sector OAB, that sector is a fraction

$$\frac{\theta}{2\pi}$$

of the whole circle, so its area is:

$$\text{area of sector } OAB = \pi\left(\frac{\theta}{2\pi}\right)$$

$$= \frac{\theta}{2}$$

What about the area of the triangle OAB? The base of the triangle is OA and has length 1. The height is DB and has height $\sin\theta$. The area of the triangle is $\frac{1}{2}\cdot$ length $\cdot$ height, hence

$$\text{area of triangle } OAB = \frac{1}{2}\sin\theta$$

Finally,

$$\text{area of triangle } OAC = \frac{1}{2}(OA)(AC)$$

$$= \frac{1}{2}\cdot 1\cdot \tan\theta$$

$$= \frac{1}{2}\tan\theta$$

Note:

$$\sin\theta = \frac{\text{opposite}}{\text{hypoteneuse}}$$

$$= \frac{DB}{OB}$$

$$= DB$$

since $OB = 1$. Thus

$$DB = \sin\theta$$

Similiarly;

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$= \frac{CA}{OA}$$

$$= CA$$

since $OA = 1$

$$\Rightarrow CA = \tan\theta$$

So

area of triangle OAB < area of sector OAB < area of triangle OAC

means:

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$\Rightarrow \sin\theta < \theta < \tan\theta$$

$$\Rightarrow \frac{1}{\sin\theta}\sin\theta < \frac{1}{\sin\theta}\theta < \frac{1}{\sin\theta}\tan\theta$$

$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$$

(take the reciprocal and use the fact that if $a < b$ then $\frac{1}{a} > \frac{1}{b}$)

or rewriting:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Now what about $-\pi/2 < \theta < 0$? The inequality still holds because $\cos\theta$ and $\sin\theta/\theta$ are even functions. This proves the theorem for $0 < |\theta| < \pi/2$

$\blacksquare$

Corollary:

$$|\sin x| \leqslant |x|$$

$\forall x \in \mathbb{R}$

Proof:

The result is obvious if $x = 0$. For $0 < |x| < \pi/2$ we use the theorem we just proved (let $\theta = x$). For $|x| \geqslant 1$ the inequality holds because $|\sin x| \leqslant 1 \ \forall x \in \mathbb{R}$. Hence

$$|\sin x| \leqslant 1 \leqslant |x|$$

if $|x| \geqslant 1$.

∎

So why do we care? These inequalities will pop up again and again in calculus. But for right now we use them to prove:

Theorem

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Proof:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \Rightarrow \operatorname{limit}_{x \to 0} \left(\frac{\sin x}{x} - 1 \right) = 0$$

$$\Rightarrow \lim_{x \to 0} \left| \frac{\sin x}{x} - 1 \right| = 0$$

Clearly

$$0 \leq \left|1 - \frac{\sin x}{x}\right| \qquad x \neq 0$$

Next; from the first theorem on page 1 we have

$$\cos x < \frac{\sin x}{x}$$

So

$$\left|1 - \frac{\sin x}{x}\right| \leq \left|1 - \cos x\right|$$

(I changed the $<$ to $\leq$. We can always relax a strict inequality in this way).

Hence:

$$0 \leq \left|1 - \frac{\sin x}{x}\right| \leq \left|1 - \cos x\right|$$

Now use the trig identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\Rightarrow 1 - \cos x = 2 \sin^2 \frac{x}{2}$$

So now we have

$$0 \leq \left|1 - \frac{\sin x}{x}\right| \leq \left|1 - \cos x\right| = \left|2 \sin^2 \frac{x}{2}\right|$$

But

$$\left|2 \sin^2 \frac{x}{2}\right| = |2| \left|\sin^2 \frac{x}{2}\right| \leq 2 \left|\sin \frac{x}{2}\right|$$

because $\left|\sin \frac{x}{2}\right| \leq 1$ and squaring a number < 1 makes it smaller.
Finally, from our corallary on page 4 we have $\left|\sin \frac{x}{2}\right| \leq |x|$

So we have

$$0 \leq \left|1 - \frac{\sin x}{x}\right| \leq |1-\cos x| = \left|2\sin^2 \frac{x}{2}\right| \leq 2\left|\sin \frac{x}{2}\right| \leq 2\left|\frac{x}{2}\right| = |x|$$

or:

$$0 \leq \left|1 - \frac{\sin x}{x}\right| \leq |x|$$

From the sandwich theorem:

$$\lim_{x \to 0} 0 \leq \lim_{x \to 0} \left|1 - \frac{\sin x}{x}\right| \leq \lim_{x \to 0} |x|$$

$$\Rightarrow 0 \leq \lim_{x \to 0} \left|1 - \frac{\sin x}{x}\right| \leq 0$$

$$\Rightarrow \lim_{x \to 0} \left|1 - \frac{\sin x}{x}\right| = 0$$

$$\Rightarrow \left|\lim_{x \to 0} \left(1 - \frac{\sin x}{x}\right)\right| = 0$$

$$\Rightarrow \lim_{x \to 0} \left(1 - \frac{\sin x}{x}\right) = 0$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{x} = 1$$

∎

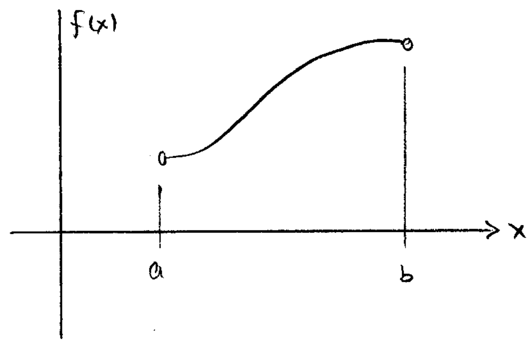
Okay, it was a big digression! But this is a useful exercise illustrating the power of inequalities in calculus.

Now onto today's subject...

Continuity at a Point

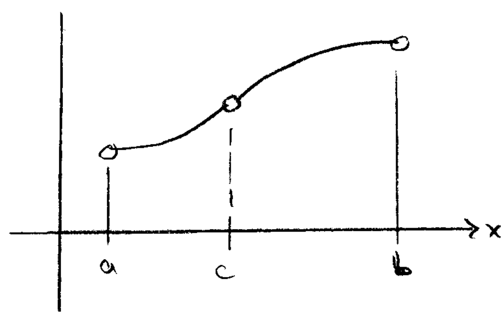
Informally, continuous functions are functions whose graph is unbroken. To tighten that argument a bit, let's restrict our function to some interval (a,b). Then we say $f(x)$ is continuous on this open interval if its graph is unbroken inside the interval, i.e., for all x such that $a < x < b$ (i.e, $x \in (a,b)$). So

(Figure A)



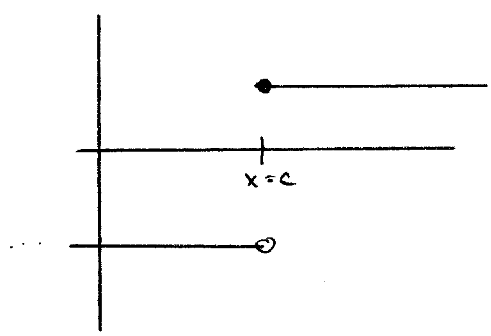
is continuous for $0 < a < b$, But

(Figure B)



is not continuous over (a,b) because it is undefined at $x=c$ (though note that it is continuous over (a,c) and (c,b)). Similiarly,

(Figure C)



is discontinuous at $x=c$.

To make this definition more precise;

Def Let $f(x)$ be defined on an open interval (a,b). The function is continuous at an interior point $x=c$ where $c \in (a,b)$ if

$$\lim_{x \to c} f(x) = f(c)$$

A function is continuous over the entire interval (a,b) if it is continuous at every point $x \in (a,c)$.

For this definition to make any sense, first the function $f(x)$ has to exist and have a finite value at $x=c$. Second the $\lim_{x\to c} f(x)$ has to exist. And finally $\lim_{x \to c} f(x)$ has to actually equal $f(c)$. So we have three conditions for continuity:

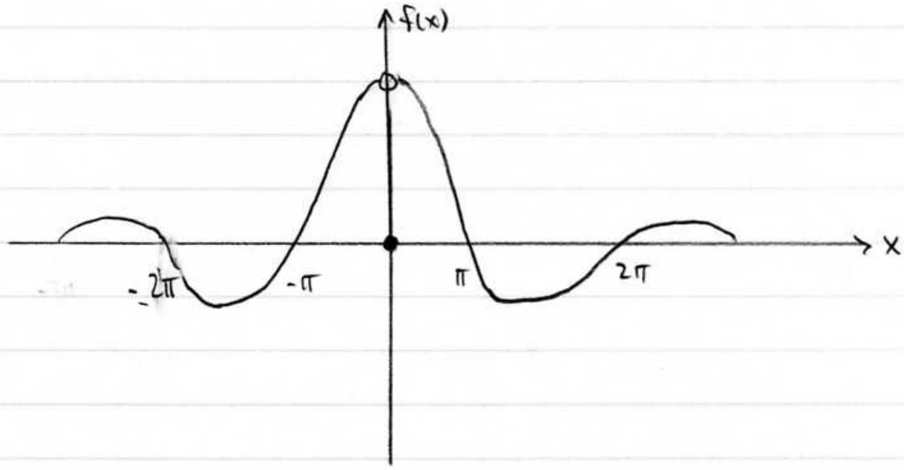
Continuity test

A function $f(x)$ is continuous at $x=c$ if and only if it meets the following three conditions;

1. $f(c)$ exists
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

Figure A on the previous page satisfies all three criteria for all $x \in (a,b)$. Figure B fails at $x=c$ because $f(c)$ does not exist, even though the $\lim_{x \to c} f(x)$ does. Figure C fails because $\lim_{x\to c} f(x)$ doesn't exist. What about

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ x = 0 & x = 0 \end{cases} \qquad ?$$



This is discontinuous at $x=0$ despite the fact that $f(0)$ exists and $\lim_{x\to 0} f(x) = 1$ exists. It fails because

$$\lim_{x\to c} f(x) = 1 \neq f(c) = 0$$

However, what about

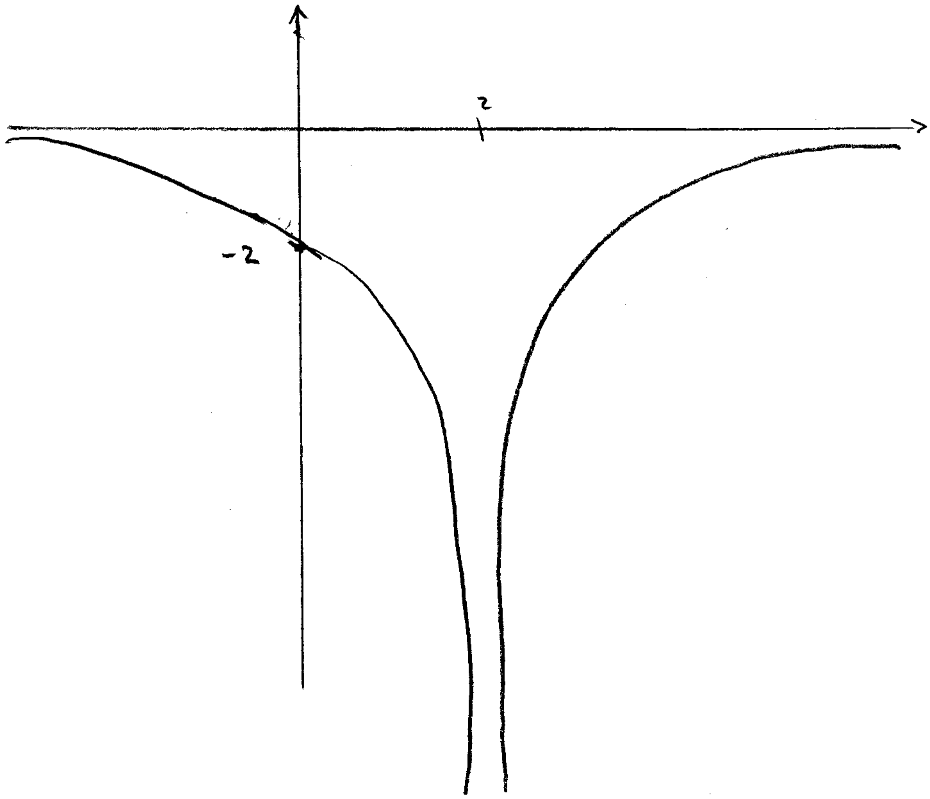
$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \quad ?$$

This function is continuous at $x=0$ and in fact $\forall\, x \in \mathbb{R}$. When we can make a function continuous by setting (or defining) its value at a point $x=c$ to be equal to its limit at $x=c$, we say the discontinuity is removeable.

Another definition, a discontinuity like in figure C is called a jump discontinuity because the left- and right-hand limits exist but they have different values.

Another kind of discontinuity is an <u>infinite discontinuity</u>. This happens when the function blows up to infinity, for example:

$$f(x) = -\frac{1}{(x-2)^2} \qquad (x \neq 0)$$



Note that $f(x) = \sin\left(\frac{1}{x}\right)$ is also discontinuous, but this time because it has no limit as $x \to 0$ (it oscillates faster and faster as $x \to 0$).

Continuous at an End-Point

The definition on page 8 stipulated that the point c was in the open interval (a,b), hence $a < c < b$. By that token,

$$f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

is continuous on $x \in (-\infty, 0)$ as well as on $x \in (0, \infty)$. The function is discontinuous at $x = 0$ because $\lim\limits_{x \to 0} f(x)$ doesn't exist. It fails.

condition 2 of the continuity test on page 8 (never mind condition 3).
But

$$\lim_{x \to 0^-} f(x) = -1$$

and

$$\lim_{x \to 0^+} f(x) = 1$$

i.e., the left- and right-hand limits do exist. The true limit fails to exist because the left- and right-hand limits are not equal. But just as we extended our limit concepts to include left- and right-hand limits, we can do the same with continuity. Specifically, we say that

$$f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

is continuous from the left at $x = 0$ since all the conditions of the continuity test now hold so long as we restrict ourselves to left-handed limits. That is:

$f(0)$ exists (equals -1)

$\lim_{x \to 0^-} f(x)$ exists (equals -1)

and

$\lim_{x \to 0^-} f(x) = f(0) = 1$

Similiarly, let

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

(I switched the equality to $x \geq 0$ from $x \leq 0$ in the previous example).

Now:

$$f(0) \text{ exists} \qquad \text{(equals 1)}$$

$$\lim_{x \to 0^+} f(x) \text{ exists} \qquad \text{(equals 1)}$$

and

$$\lim_{x \to 0} f(x) = f(0) = 1$$

We say $f(x)$ is <u>continuous from the right</u>. If you want a formal definition:

> **Def** A function f is <u>continuous at a left endpoint</u> $x=a$ of its domain if
>
> $$\lim_{x \to a^+} f(x) = f(a)$$
>
> and is continuous at a right endpoint $x=b$ of its domain if
>
> $$\lim_{x \to b^-} f(x) = f(b)$$

Rules of Continuity

> **Theorem:** If f and g are continuous at $x=c$, then the following are also continuous at $x=c$
>
> 1. $f+g$ and $f-g$
> 2. fg
> 3. kf where $k \in \mathbb{R}$
> 4. f/g provided $g(c) \neq 0$
> 5. $[f(x)]^{m/n}$ provided that $[f(x)]^{m/n}$ is defined on an open interval containing c and $m, n \in \mathbb{Z}$.

And one straightforward extention:

Theorem: (Continuity of Composites) If $f(x)$ is continuous at $x = c$ and $g(y)$ is continuous at $y = f(c)$, then

$$(g \circ f)(x)$$

is continuous at $x = c$.

Example

Let

$$f(x) = \sin x$$

$$g(x) = \frac{1}{x}$$

Then:

$$(g \circ f)(x) = \frac{1}{\sin x}$$

Now $\sin x$ is continuous $\forall\ x \in \mathbb{R}$ and $\frac{1}{x}$ is continuous $\forall\ x \neq 0$. So

$$(g \circ f)(x) = \frac{1}{\sin x}$$

is continuous except where the argument of g is non-zero. this is where $x \neq n\pi$, $n \in \mathbb{Z}$.

Example

$$f(x) = \cos^2 x$$

$$g(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Since $f(x) \geq 0$ always, then

$$(g \circ f)(x) = 1 \quad \forall \; x \in \mathbb{R}$$

and hence the composite function is continuous everywhere.

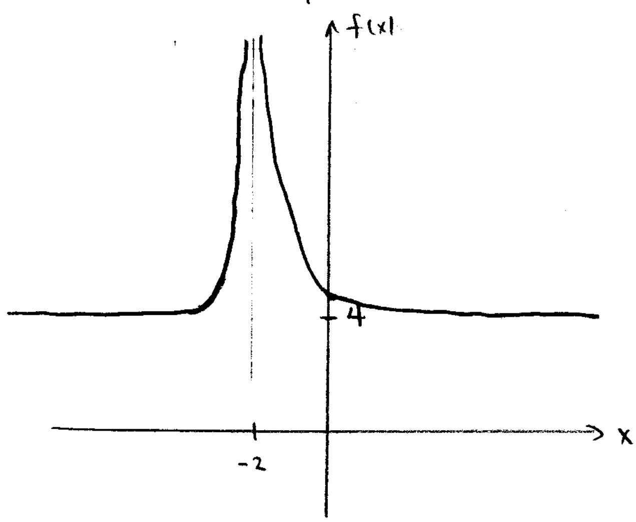
Class exercize

(#14, p. 95)

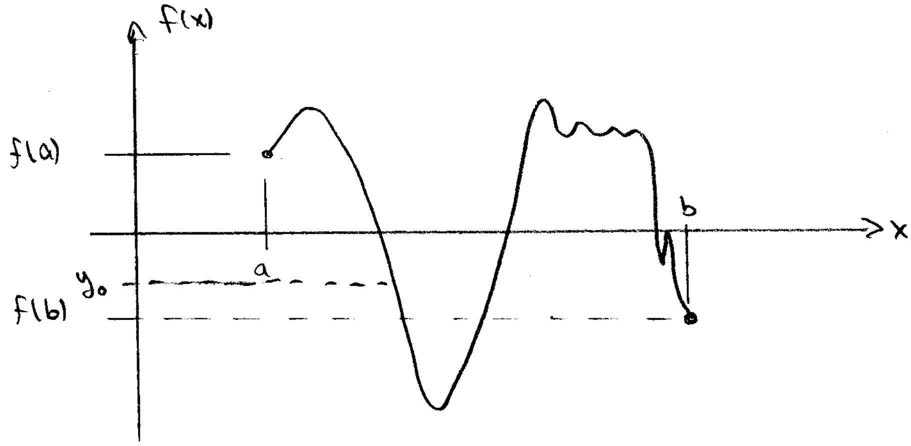
At what points is $y = \dfrac{1}{(x+2)^2} + 4$ continuous?

Solution:

$y(x)$ is undefined at $x = -2$, but for all other points it passes all the criteria of the continuity test. So it is continuous $\forall \; x \in \mathbb{R}$ except $x = -2$.



The Intermediate Value Theorem



Let $f(x)$ be continuous on $[a,b]$ (so it is continuous in the interior (a,b) and is right-continuous at $x=a$ and left continuous at $x=b$). Choose any value y_0 such that $f(b) \leq y_0 \leq f(a)$ in this example. Since $f(x)$ is continuous and hence the graph is unbroken, it is obvious that $f(x)$ has to equal the value y_0 somewhere in the interval $[a,b]$. This is clearly true for all y_0 that lie between $f(a)$ and $f(b)$. Hence:

> **Theorem** (The Intermediate Value Theorem) Suppose $f(x)$ is continuous on an interval I and let $a < b$ be two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, there exists (i.e., always is) a number c such that $a \leq c \leq b$ such that $f(c) = y_0$.

Example

Show that the function $f(x) = 2x^3 - 4x^2 + 5x - 4$ has at least one zero between $x = 1$ and $x = 2$.

Solution:

$f(x) = 2x^3 - 4x^2 + 5x - 4$ is continuous $\forall\ x$ and hence for $1 \leq x \leq 2$. Now;

$$f(1) = 2 - 4 + 5 - 4 = -1$$

and

$$\begin{aligned} f(2) &= 2 \cdot 8 - 4 \cdot 4 + 5 \cdot 2 - 4 \\ &= 16 - 16 + 10 - 4 \\ &= 6 \end{aligned}$$

Since $f(1) = -1 \leq 0 \leq 6 = f(2)$ and $f(x)$ is continuous, then by the intermediate value theorem there is some point x where $-1 \leq x \leq 6$ such that $f(x) = 0$.

Miscellaneous Limit Problems to Tease Your Brain

Example

(# 29, p. 96) Find $\lim\limits_{x \to \pi} \sin(x - \sin x)$

Soln:

$$\lim_{x \to \pi} x - \sin x = \pi - 0 = \pi$$

and

$$\lim_{y \to \pi} \sin y = 0$$

$$\Rightarrow \lim_{x \to \pi} \sin(x - \sin x) = 0$$

Example

Find $\lim_{x \to \pi} \cos(x - \sin x)$

Solution:

$$\lim_{x \to \pi} x - \sin x = \pi - \sin \pi = \pi$$

$$\lim_{y \to \pi} \cos y = \cos \pi = -1$$

$$\Rightarrow \lim_{x \to \pi} \cos(x - \sin x) = -1$$

Note:

$$\lim_{x \to x_0} (g \circ f)(x) = \lim_{y \to \lim_{x \to x_0} f(x)} g(y)$$

Example

Find $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ (Hint: Multiply top & bottom by the conjugate of the numerator).

Solution:

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \underbrace{\lim_{x \to 0} \left(\frac{1}{1+\cos x}\right)}_{\frac{1}{2}}$$

$$= \frac{1}{2} \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2$$

$$= \frac{1}{2} \left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2$$

$$= \frac{1}{2}$$

since $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (see page 4 of these notes).

<u>Example</u>

Find $\lim_{x \to 0} \frac{1 - \cos x}{x}$

Solution:

Same idea as above: $\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x} \left(\frac{1+\cos x}{1+\cos x}\right)$

$$= \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)\left(\lim_{x \to 0} \sin x\right)\left(\lim_{x \to 0} \frac{1}{1+\cos x}\right)$$

$$= 1 \cdot 0 \cdot \frac{1}{2} = 0$$