APPM 1350: Section 2.2: Differentiation Rules

By definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

In principle we can always compute the derivative this way, but as we have seen it can be very tedious. Fortunately, and remarkably, we can construct rules for simplifying such calculations under certain circumstances. We now explore some of these rules. Memorize these rules!

Derivative of a Constant

$$\boxed{\text{If } c \text{ is a constant then } \frac{d}{dx} c = 0}$$

Proof:

$$\frac{d}{dx} c = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

Example

$$\frac{d}{dx} 3 = 0$$

$$\frac{d}{dx} \left(\frac{1}{5}\right)^{300/4.2} = 0$$

$$\frac{d}{dx} \cos\frac{\pi}{8} = 0$$

Power Rule for Positive Integers

$$\boxed{\text{If } n \text{ is a positive integer, then } \frac{d}{dx} x^n = nx^{n-1}}$$

I highly recommend looking at the proof on page 121 of the book. I am not going to repeat it here. But let's see if the rule actually works by trying an example.

Example

Show $\frac{d}{dx} x^3 = 3x^2$ using the definition of the derivative.

Solution:

$$\frac{d}{dx} x^3 = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2$$

Note, #53 on page 130 is a homework problem that asks you to prove this theorem in a slightly different way than done in the book on page 121.

Constant Multiple Rule

> If $f(x)$ is a differentiable function of x and c is a constant, then
>
> $$\frac{d}{dx} c f(x) = c \frac{d}{dx} f(x)$$

The proof of this is trivial:

$$\frac{d}{dx} c f(x) = \lim_{h \to 0} \frac{c f(x+h) - c f(x)}{h}$$

$$= \lim_{h \to 0} c \left[\frac{f(x+h) - f(x)}{h} \right]$$

$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= c \frac{d}{dx} f(x)$$

$\blacksquare$

Example

$$\frac{d}{dx} 3x^4 = 3 \frac{d}{dx} x^4$$

$$= 3(4x^3)$$

$$= 12x^3$$

$$\frac{d}{dx} \left(\cos \frac{\pi}{8}\right) x^{200} = \left(\cos \frac{\pi}{8}\right) \frac{d}{dx} x^{200}$$

$$= (200)\left(\cos \frac{\pi}{8}\right) x^{199}$$

Note the corollary:

$$\boxed{\begin{aligned}\frac{d}{dx}\left[-f(x)\right] &= \frac{d}{dx}(-1)\left[f(x)\right] \\ &= -\frac{d}{dx}f(x)\end{aligned}}$$

The Sum Rule

If $f(x)$ and $g(x)$ are differentiable functions of x, then their sum $f(x)+g(x)$ is differentiable at every point x where they are both differentiable and

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

This is another trivial theorem to prove, see page 123 of the book.

Note that this rule holds for subtracting functions as well as adding them.

Example

Find $\frac{d}{dx}\, 3x^3 + 4(\ln 3)\,x^2$

Solution:

$$\frac{d}{dx}\left[3x^3 + (4\ln 3)\,x^2\right] = 3\frac{d}{dx}x^3 + 4\ln 3\,\frac{d}{dx}x^2$$

$$= 3(3x^2) + (4\ln 3)\,2x$$

$$= 9x^2 + (8\ln 3)\,x$$

Product Rule

If $f(x)$ and $g(x)$ are differentiable at x, then so is their product and

$$\frac{d}{dx}\left[f(x)\,g(x)\right] = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]$$

A more common way of writing this is:

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

The proof is easy, but a bit messy. See p. 125 of the book

Example

Find $\frac{d}{dx}(x+4)(x^2-8)$

Solution:

$$\frac{d}{dx}(x+4)(x^2-8) = \left[\frac{d}{dx}(x+4)\right](x^2-8) + (x+4)\left[\frac{d}{dx}(x^2-8)\right]$$

$$= (1)(x^2-8) + (x+4)(2x)$$

$$= x^2 - 8 + 2x^2 + 8x$$

$$= 3x^2 + 8x - 8$$

Quotient Rule

If $f(x)$ and $g(x)$ are differentiable at x and $g(x) \neq 0$, then the quotient $f(x)/g(x)$ is differentiable at x and

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

See page 126 for a proof.

Example

Find $\displaystyle\frac{d}{dx}\frac{x^2-8}{x^4+3x}$ $\qquad [x \neq 0 \text{ and } x \neq -(3)^{1/3}]$

Solution:

$$\frac{d}{dx}\frac{x^2-8}{x^4+3x} = \frac{(x^4+3x)\frac{d}{dx}(x^2-8) - (x^2-8)\frac{d}{dx}(x^4+3x)}{(x^4+3x)^2}$$

$$= \frac{(x^4+3x)(2x) - (x^2-8)(4x^3+3)}{(x^4+3x)^2}$$

This can obviously be simplified, but I am going to leave it in the form.

Power Rule for Negative Integers

> If n is a negative integer and $x \neq 0$, then
>
> $$\frac{d}{dx} x^n = n x^{n-1}$$

This follows from the quotient theorem. See p. 127 for the proof.

Example

Find $\frac{d}{dx}\left(x + \frac{1}{x^2}\right)$ $\qquad (x \neq 0)$

Solution:

$$\begin{aligned}
\frac{d}{dx}\left(x + \frac{1}{x^2}\right) &= \frac{d}{dx}\left(x + x^{-2}\right) \\
&= \frac{d}{dx} x + \frac{d}{dx} x^{-2} \\
&= 1 - 2x^{-3} \\
&= 1 - \frac{2}{x^3}
\end{aligned}$$

Higher Order Derivatives

The derivative $f'(x)$ of $f(x)$ is called the first order derivative. Often $f'(x)$ is itself is differentrable. In that case:

$$f''(x) = \frac{d}{dx} f'(x)$$

$$= \frac{d}{dx}\left[\frac{d}{dx} f(x)\right]$$

$$= \frac{d^2}{dx^2} f(x)$$

is called the <u>second derivative</u> of $f(x)$ at x. If this is differentrable we can in turn differentrate it to get the <u>third derivative</u>

$$f'''(x) = \frac{d}{dx} f''(x)$$

$$= \frac{d}{dx}\left[\frac{d}{dx}\left(\frac{d}{dx} f\right)\right]$$

$$= \frac{d^3}{dx^3} f(x)$$

In general we denote the <u>nth derivative</u> (or <u>nth order derivative</u>) as:

$$y^{(n)} = \frac{d^n}{dx^n} f(x)$$

$$= \frac{d}{dx}\left[\frac{d^{n-1}}{dx^n} f(x)\right]$$

$$= \frac{d}{dx} f^{(n-1)}(x)$$

Example

Find derivatives of all orders of $f(x) = 3x^3 + 4x^2 + 2$

Solution:

$$f'(x) = \frac{d}{dx} f(x) = 9x^2 + 8x$$

$$f''(x) = \frac{d^2}{dx^2} f(x) = 18x + 8$$

$$f'''(x) = \frac{d^3}{dx^3} f(x) = 18$$

$$f^{(4)}(x) = \frac{d^4}{dx^4} f(x) = 0$$

All other derivatives of higher order are equal to zero. So

$$f^{(n)}(x) = \frac{d^n}{dx^n} f(x) = 0 \quad \text{for} \quad n \geq 4$$

Summary of the Rules

Assume f, g are differentiable at x. Then

1. $\frac{d}{dx} c = 0$ $\quad c = \text{constant}$
2. $\frac{d}{dx} x^n = n x^{n-1}$ $\quad n \in \mathbb{Z}$ and $n \neq 0$
3. $\frac{d}{dx} cf = c \frac{d}{dx} f$ $\quad c = \text{constant}$
4. $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$
5. $\frac{d}{dx} fg = g \frac{df}{dx} + f \frac{dg}{dx}$
6. $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \frac{df}{dx} - f \frac{dg}{dx}}{g^2}$ $\quad (g \neq 0)$

More complicated examples

Example

(#11, p.129)

$$\frac{d}{ds}\left(\frac{1}{3s^2} - \frac{5}{2s}\right) = \frac{1}{3}\frac{d}{ds}s^{-2} - \frac{5}{2}\frac{d}{ds}s^{-1}$$

$$= \frac{1}{3}(-2s^{-3}) - \frac{5}{2}(-s^{-2})$$

$$= -\frac{2}{3s^3} + \frac{5}{2s^2}$$

Example

(#22, p.129)

$$\frac{d}{dx}(2x-7)^{-1}(x+5) = \frac{d}{dx}\frac{x+5}{2x-7}$$

$$= \frac{(2x-7)\frac{d}{dx}(x+5) - (x+5)\frac{d}{dx}(2x-7)}{(2x-7)^2}$$

$$= \frac{(2x-7)(1) - (x+5)(2)}{(2x-7)^2}$$

$$= \frac{2x-7-2x-10}{(2x-7)^2}$$

$$= -\frac{17}{(2x-7)^2}$$

using the quotient rule.

Example
(# 24, p. 129)

Find $\frac{du}{dx}$ for $u = \frac{5x+1}{2\sqrt{x}}$

Solution:

You will need the result $\boxed{\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \qquad (x \neq 0)}$ which

we derived in previous lectures. So now, let

$$f(x) = 5x+1 \qquad \Rightarrow \qquad f'(x) = 5$$
$$g(x) = 2\sqrt{x} \qquad\qquad g'(x) = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}} = x^{-1/2}$$

and

$$\frac{d}{dx}\,\frac{5x+1}{2\sqrt{x}} = \frac{d}{dx}\left(\frac{f}{g}\right)$$

$$= \frac{gf' - fg'}{g^2}$$

$$= \frac{(2\sqrt{x})(5) - (5x+1)(x^{-1/2})}{4x}$$

$$= \frac{10\sqrt{x} - (5x-1)\left(\frac{1}{\sqrt{x}}\right)}{4x}$$

$$= \frac{\sqrt{x}}{\sqrt{x}}\;\frac{10\sqrt{x} - (5x-1)\left(\frac{1}{\sqrt{x}}\right)}{4x}$$

$$= \frac{10x - 5x + 1}{4x^{3/2}}$$

$$= \frac{5x+1}{4x^{3/2}}$$

We could also write this as $\frac{5}{4}x^{-1/2} + \frac{1}{4}x^{-3/2}$