APPM 1350: Section 2.5 : The Chain Rule

Problems such as finding

$$\frac{d}{dx}\left(1+3x^2\right)^4$$

involve finding the derivative of a composite function $(f \circ g)(x)$. In this example $f(u) = u^4$ and $g(x) = 1+3x^2$. Up until now to do this problem you would have to repeatedly use the product rule or you would have to expand $(1+3x^2)^4$ as a polynomial. Both are valid approaches, but fortunately there is an even easier way.

The Chain Rule

Theorem: If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then $(f \circ g)(x) = f[g(x)]$ is differentiable at x and

$$(f \circ g)'(x) = f'[g(x)]\; g'(x)$$

or in terms of Leibniz' notation, let $y = f(u)$ and $u = g(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

where dy/du is evaluated at $u = g(x)$

So consider our example again. If $f(u) = u^4$ (the "outer" function) and $g(x) = 1+3x^2$ (the "inner" function), then

$$(f \circ g)(x) = \underbrace{f[\underbrace{g(x)}_{\text{inner function}}]}_{\text{outer function}} = (1+3x^2)^4$$

Then

$$\frac{d}{dx}(f \circ g)(x) = \underbrace{\left.\frac{df}{du}\right|}_{\substack{\text{evaluated} \\ \text{at } u=g(x)}} \cdot \underbrace{\left.\frac{dg}{dx}\right|}_{\substack{\text{evaluated} \\ \text{at } x}}$$

$$= \left. 4u^3 \right|_{u=1+3x^2} \cdot 6x$$

$$= 4(1+3x^2)^3 \, 6x$$

$$= 24x(1+3x^2)^3$$

The "Outside-Inside" Rule

So when we write $(f \circ g)(x)$, think of $f(u)$ as the "outside" function and $g(x)$ as the "inside" function. Then

$$\boxed{\begin{aligned} \frac{d}{dx}(f \circ g)(x) &= \left[\frac{d}{du}\text{"outside" evaluated at } u=g(x)\right]\left[\frac{d}{dx}\text{"inside"}\right] \\ &= \left.\frac{df}{du}\right|_{u=g(x)} \cdot \frac{dg}{dx} \end{aligned}}$$

Example

$$\frac{d}{dx}(3x+2)^2 = \underbrace{2(3x+2)^1}_{\text{outer}}\ \underbrace{(3)}_{\text{inner}}$$

$$= 6(3x+2)$$

$$\begin{cases} f(u) = u^2 \\ g(x) = 3x+2 \\ (f \circ g)(x) = (3x+2)^2 \end{cases}$$

Example

$$\frac{d}{dx}\sin(x^2) = \underbrace{\cos(x^2)}_{\text{outer}}\ \underbrace{2x}_{\text{inner}}$$

$$= 2x\cos(x^2)$$

$$\begin{cases} f(u) = \sin u \\ g(x) = x^2 \\ (f \circ g)(x) = \sin(x^2) \end{cases}$$

Example
(#7, p.161)

$$\frac{d}{dx}\left(1 - \frac{x}{7}\right)^{-7} = \underbrace{-7\left(1 - \frac{x}{7}\right)^{-8}}_{\text{outer}}\ \underbrace{\left(-\frac{1}{7}\right)}_{\text{inner}}$$

$$= \left(1 - \frac{x}{7}\right)^{-8}$$

$$\begin{cases} f(u) = u^{-7} \\ g(x) = 1 - \frac{x}{7} \\ (f \circ g)(x) = \left(1 - \frac{x}{7}\right)^{-7} \end{cases}$$

Power Chain Rule

This just formally states a rule we have been using above.

> If $u(x)$ is a differentiable function and n is an integer, then u^n is differentiable and
>
> $$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

Example

$$\frac{d}{dx}\cos^5 x = \frac{d}{dx}\underbrace{(\cos x)}_{u}^{5}$$

$$= \underbrace{5(\cos x)^4}_{n u^{n-1}} \underbrace{(-\sin x)}_{\frac{du}{dx}}$$

$$= -5 \sin x \cos^4 x$$

$$\left\{ \begin{array}{l} f(u) = u^5 \\ g(x) = \cos x \\ (f \circ g)(x) = \cos^5 x \end{array} \right.$$

Lots More Examples

Example

(#26, p.161)

$$\frac{d}{dx}\left(\frac{1}{x}\sin^{-5}x - \frac{x}{3}\cos^3 x\right) = \frac{d}{dx}\frac{1}{x}\sin^{-5}x - \frac{d}{dx}\frac{x}{3}\cos^3 x$$

$$= \sin^{-5}x \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\frac{d}{dx}\sin^{-5}x$$

$$- \cos^3 x \frac{d}{dx}\left(\frac{x}{3}\right) - \frac{x}{3}\frac{d}{dx}\cos^3 x$$

product rule

$$= \sin^{-5}x \frac{d}{dx}x^{-1} + \frac{1}{x}\frac{d}{dx}(\sin x)^{-5}$$

$$- \frac{1}{3}\cos^3 x \frac{d}{dx}x - \frac{x}{3}\frac{d}{dx}(\cos x)^3$$

$$= \sin^{-5}x(-x^{-2}) - \frac{1}{x}5(\sin x)^{-6}\cos x$$

$$- \frac{1}{3}\cos^3 x \cdot 1 - \frac{x}{3}3(\cos x)^2(-\sin x)$$

$$= -\frac{\sin^{-5}x}{x^2} - \frac{5}{x}\sin^{-6}x\cos x$$

$$- \frac{1}{3}\cos^3 x + x\sin x\cos^2 x$$

Example

(#44, p.161)

$$\frac{d}{dt}\cos\left(5\sin\left(\frac{t}{3}\right)\right) = \underbrace{-\sin\left(5\sin\frac{t}{3}\right)}_{\text{outer}}\ \underbrace{\frac{d}{dx}\,5\sin\frac{t}{3}}_{\text{inner}}$$

$$= -\sin\left(5\sin\frac{t}{3}\right)\left[5\ \underbrace{\cos\frac{t}{3}}_{\text{outer}}\ \underbrace{\frac{d}{dt}\left(\frac{t}{3}\right)}_{\text{inner}}\right]$$

$$= -5\sin\left(5\sin\frac{t}{3}\right)\left(\cos\frac{t}{3}\right)\left(\frac{1}{3}\right)$$

$$= -\frac{5}{3}\sin\left(5\sin\frac{t}{3}\right)\cos\left(\frac{t}{3}\right)$$

Example

$$\frac{d}{dx}\sqrt{3x^2-5} = \frac{d}{dx}\left(3x^2-5\right)^{\frac{1}{2}}$$

$$= \underbrace{\frac{1}{2}\left(3x^2-5\right)^{-1/2}}_{\text{outer}}\ \underbrace{(6x)}_{\text{inner}}$$

using $\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}}$

$= \frac{1}{2}u^{-1/2}$

$$= \frac{3x}{\sqrt{3x^2-5}}$$

Example

$$\frac{d}{dx}\frac{1}{(3x^2+5)^4} = \frac{d}{dx}(3x^2+5)^{-4}$$

$$= \underbrace{-4(3x^2+5)^{-5}}_{\text{outer}}\ \underbrace{(6x)}_{\text{inner}}$$

$$= -24\frac{x}{(3x^2+5)^5}$$

Example

$$\frac{d}{dx}\left(\frac{x^2-2}{2x^2+1}\right)^2 = \underbrace{2\left(\frac{x^2-2}{2x^2+1}\right)^1}_{\text{outer}}\ \underbrace{\frac{d}{dx}\left(\frac{x^2-2}{2x^2+1}\right)}_{\text{inner}}$$

$$= 2\left(\frac{x^2-2}{2x^2+1}\right)\frac{(2x^2+1)\frac{d}{dx}(x^2-2) - (x^2-2)\frac{d}{dx}(2x^2+1}{(2x^2+1)^2}$$

$$= \frac{2(x^2-2)\left[(2x^2+1)(2x) - (x^2-2)(4x)\right]}{(2x^2+1)^3}$$

$$= \frac{4x(x^2-2)\left[\cancel{2x^2}+1 - \cancel{2x^2}+4\right]}{(2x^2+1)^3}$$

$$= \frac{20x(x^2-2)}{(2x^2+1)^3}$$

Example

$$\frac{d^2}{dx^2}\cos(3x^2+4) = \frac{d}{dx}\left\{\frac{d}{dx}\cos(3x^2+4)\right\}$$

$$= \frac{d}{dx}\left\{\underbrace{-\sin(3x^2+4)}_{\text{outer}}\ \underbrace{\frac{d}{dx}(3x^2+4)}_{\text{inner}}\right\}$$

$$= \frac{d}{dx}\left\{-\sin(3x^2+4)\ (6x)\right\}$$

$$= -6\frac{d}{dx}\ x\sin(3x^2+4)$$

$$= -6\left\{\sin(3x^2+4)\frac{d}{dx}x + x\frac{d}{dx}\sin(3x^2+4)\right\}$$

$$= -6\left\{\sin(3x^2+4) + x\left[\underbrace{\cos(3x^2+4)}_{\text{outer}}\ \underbrace{\frac{d}{dx}(3x^2+4)}_{\text{inner}}\right]\right\}$$

$$= -6\left\{\sin(3x^2+4) + x\left[\cos(3x^2+4)\ (6x)\right]\right\}$$

$$= -6\sin(3x^2+4) - 36x^2\cos(3x^2+4)$$