APPM 1350: Section 2.7: Related Rates of Change

Today's lecture is dedicated to doing interesting things with calculus. In particular we discuss related rates. Many systems in science and engineering are parameterized by two variables that are related by some equation. If these variables are functions of time, then using implicit differentiation we can ask how fast one variable changs as the other one changes. To explore this idea we will do a bunch of examples.

Example

The heart is to a first approximation a sphere of radius r. The volum of the left ventricle is thus

$$V_h = \frac{4}{3}\pi r^3$$

During systole the left ventricle contracts and blood is pumped out of the ventricle through the aorta into the arterial system of the body. The rate at which the volume of the heart diminishes as the radius of the heart diminishes (due to contraction) is given by differentiating the equation above as a function of time:

$$V_h = \frac{4}{3}\pi r^3$$

$$\Rightarrow \frac{dV_h}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \frac{dV_h}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right)$$

$$\Rightarrow \frac{dV_h}{dt} = 4\pi r^2\frac{dr}{dt}$$

This is an example of a <u>related rate</u>, in this case how $\frac{dV_h}{dt}$ is related to $\frac{dr}{dt}$. From this we can say all sorts of interesting things about cardiac physiology. For example ...

Since blood is largely incompressible, the rate at which the volume of blood decreases in the heart is equal to the negative of the rate at which blood is pumped through the aorta in to the body. This is called the cardiac output. Hence:

$$\text{Cardiac output} = -\frac{dV_h}{dt} = -4\pi r^2 \frac{dr}{dt} \qquad (1)$$

The cardiac output of the average sedantary man aged 20 to 35 years is 5.8 L/min at rest and 9.8 L/min during exercize. The mean volume of the heart before contraction in these men is 780 mL. So

$$V_h = 780\,\text{mL} = \frac{4}{3}\pi r^3$$

$$\Rightarrow r = \left[\frac{3}{4\pi}(780\,\text{mL})\right]^{1/3}$$

$$= \left[\frac{3}{4\pi} 780\,\text{cm}^3\right]^{1/3} \qquad \text{since } 1\,\text{mL} = 1\,\text{cm}^3$$

$$= 5.7\,\text{cm}$$

Now, $\frac{dr}{dt}$ is the velocity of the heart wall during contraction. So what is that velocity at rest and during exercize when the heart just starts contracting? We get this from rearranging (1):

$$\frac{dr}{dt} = -\frac{\text{cardiac output}}{4\pi r^2}$$

At rest:

$$\frac{dr}{dt} = -\frac{5800\,\text{mL/min}}{4\pi(5.7\,\text{cm})^2}$$

$$= -\frac{5800\,\frac{\text{cm}^3}{\text{min}}}{4\pi(5.7\,\text{cm})^2}$$

$$= -14.2\,\frac{\text{cm}}{\text{min}}$$

and during exercise

$$\frac{dr}{dt} = -\frac{9800\ ^{mL}/_{min}}{4\pi\,(5.7\text{ cm})^2}$$

$$= -\frac{9800\ ^{cm^3}/_{min}}{4\pi\,(5.7\text{cm})^2}$$

$$= -24\,\frac{\text{cm}}{\text{min}}$$

The velocity of the heart wall during contraction is an important diagnostic indicator of disease and can be measured using ultrasound or MRI. What we just derived are the norms for a healthy male couch potato. If the wall velocity is significantly smaller than this than it may suggest heart failure or ischemic heart disease. Alternately, from (1) we can compute the cardiac output by measuring the radius of the heart and the wall velocity. This is essentially how this is really done in cardiology, all based on implicit differentiation!

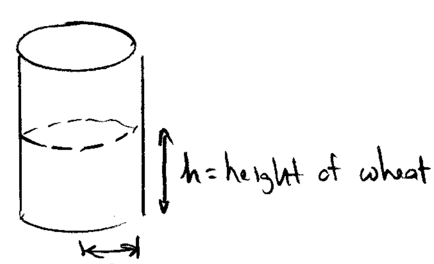
<u>Example</u>

A cylindrical tank of radius 10 feet is being filled with wheat at the rate of 314 ft^3/min. How fast is the depth of the wheat increasing?

Solution:

The volume of a cylindrical tank is

$$V = \pi r^2 h$$



where r is the radius and h is the height. In our case the tank merely constrains the wheat to be in a cylindrical shape where r is the radius of the tank (which is constant) and h is the height of the wheat.

Thus:

$$\frac{d}{dt} V = \frac{d}{dt} \pi r^2 h$$

$$\Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \qquad \text{since } r = \text{constant}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{\pi r^2} \frac{dV}{dt}$$

If $r = 10\text{ ft}$ and $\frac{dV}{dt} = 314 \text{ ft}^3/\text{min}$, then:

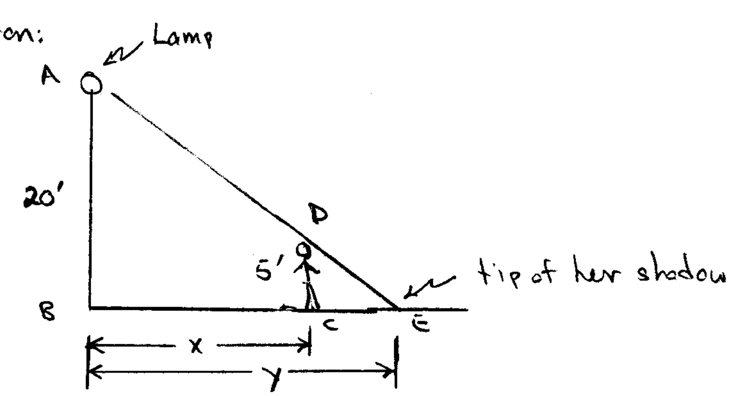
$$\frac{dh}{dt} = \frac{1}{\pi (10\text{ft})^2} \, 314 \frac{\text{ft}^3}{\text{min}}$$

$$= 3.14 \frac{\text{ft}}{\text{min}}$$

Example

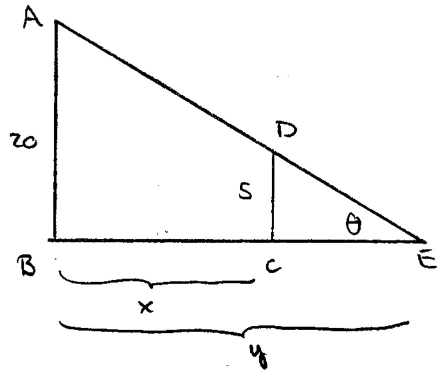
A 5-foot girl is walking toward a 20-foot lamppost at a rate of 6 ft/sec. How fast is the tip of her shadow (cast by the lamp) moving?

Solution:



First draw a diagram! Label the distance from the base of the lamppost to the girl as the distance x and the distance to the tip of her shadow y. Label points on the picture points A, B, C, D, E.

Next, note that we get two similiar triangles from this geometry.



Now derive an equation relating x and y.

Note that

$$\tan\theta = \frac{CD}{CE} = \frac{5}{y-x} \qquad (2)$$

and

$$\tan\theta = \frac{AB}{BE} = \frac{20}{y} \qquad (3)$$

Equating (2) and (3) we have:

$$\frac{5}{y-x} = \frac{20}{y}$$

$$\Rightarrow 5y = 20(y-x)$$

$$\Rightarrow y = 4(y-x)$$

$$\Rightarrow 3y = 4x$$

$$\Rightarrow 3\frac{dy}{dt} = 4\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{4}{3}\frac{dx}{dt}$$

The girl is walking toward the lamppost at 6 ft/sec, hence

$$\frac{dx}{dt} = -6 \frac{ft}{sec}$$

and thus the velocity of her shadow is:

$$\frac{dy}{dt} = \frac{4}{3}\frac{dx}{dt}$$

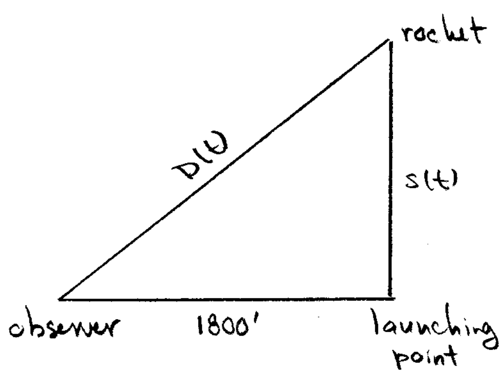
$$= \frac{4}{3}\left(-6 \frac{ft}{sec}\right)$$

$$= -8 \frac{ft}{sec}$$

Example

A rocket is shot vertically upward with an initial velocity of 400 ft/sec. Its height s after t seconds is $s = 400t - 16t^2$ feet. How fast is the distance changing between the rocket and an observer on the ground 1800 feet away from the launching site when the rocket is still rising and is 2400 feet above the ground?

Solution:



The distance $D(t)$ between the observer and rocket at time t is given by

$$D^2 = (1800)^2 + s^2$$

Hence

$$\frac{d}{dt} D^2 = \frac{d}{dt}\left[(1800)^2 + s^2\right]$$

$$\Rightarrow 2D\frac{dD}{dt} = 2s\frac{ds}{dt}$$

$$\Rightarrow \frac{dD}{dt} = \frac{s}{D}\frac{ds}{dt} \qquad (4)$$

We are interested in $\frac{dD}{dt}$ when $s = 2400$ feet. The value of D at that height is:

$$D = \sqrt{(1800)^2 + (2400)^2}$$

$$= 3000 \text{ ft}$$

Now what is $\frac{ds}{dt}$? Since $s = 400t - 16t^2$,

$$\frac{ds}{dt} = 400 - 32t$$

This is the velocity at time t. What is t when $s = 2400$ ft? Solve

$$2400 = 400t - 16t^2$$

$$150 = 25t - t^2$$

$$\Rightarrow t^2 - 25t + 150 = 0$$

$$\Rightarrow (t-10)(t-15) = 0$$

The rocket reaches this height at 10 sec on the way up and 15 sec on the way down. So in our case we are interested in the rocket going up, so $t = 10$ sec. Thus...

$$\frac{ds}{dt}(t = 10\,sec) = 400 - 32(10)$$

$$= 400 - 320$$

$$= 80 \ ^{ft}/_{sec}$$

and returning to (4) we have

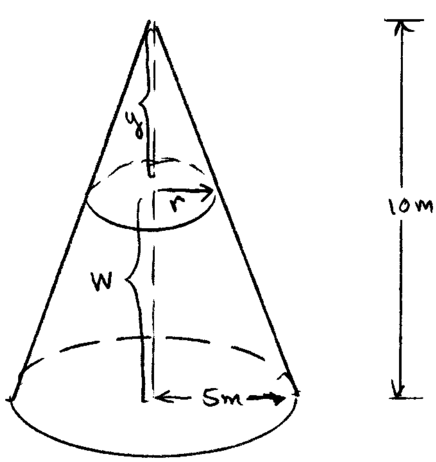
$$\frac{dD}{dt} = \frac{2400\,ft}{3000\,ft}\left(80\,\frac{ft}{sec}\right)$$

$$= 64\,\frac{ft}{sec}$$

Example

Water is pouring into an inverted cone at the rate of 3.14 $^{m^3}/_{min}$. The height of the cone is 10 m and the radius of its base is 5 m. How fast is the water level rising when the water stands 7.5 m above the base?

Solution:



Let w be the height of the water. Let y be the distance from the top of the water to the top of the cone. Note that $y = 10 - w$. What we want to know is $\frac{dw}{dt}$ when $w = 7.5$ m given $\frac{dV_{water}}{dt} = 3.14\,\frac{m^3}{min}$

The volume of a cone with base radius r and height h is.

$$V = \frac{1}{3}\pi r^2 h$$

So the total volume of our cone (radius 5m and height 10m) is:

$$V_{cone} = \frac{1}{3}\pi (5m)^2(10m)$$

$$= \frac{1}{3}\pi\, 250\ m^3$$

The air above the water line is in a cone of base radius r and height y. Hence its volume is:

$$V_{air} = \frac{1}{3}\pi r^2 y$$

The volume of the water is the difference between these two:

$$V_{water} = V_{cone} - V_{air}$$

$$= \frac{1}{3}\pi\, 250 - \frac{1}{3}\pi r^2 y$$

By similiar triangles;

$$\frac{5}{10} = \frac{r}{y}$$

$$\Rightarrow r = \frac{y}{2}$$

hence

$$V_{water} = \frac{1}{3}\pi\, 250 - \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y$$

$$= \frac{250}{3}\pi - \frac{\pi}{12}y^3$$

$$= \frac{250}{3}\pi - \frac{\pi}{12}(10-w)^3 \qquad \text{since } y = 10 - w$$

The rate at which the water is increasing is $\frac{dV_{water}}{dt}$. This equation relates V_{water} to w and what we want to know is $\frac{dw}{dt}$. So differentiate both sides:

$$\frac{d}{dt} V_{water} = \frac{d}{dt}\left[\frac{250\pi}{3} - \frac{\pi}{12}(10-w)^3\right]$$

$$\Rightarrow \frac{dV_{water}}{dt} = -\frac{\pi}{12}\frac{d}{dt}(10-w)^3$$

$$= -\frac{\pi}{12} 3(10-w)^2 \frac{d}{dt}(10-w)$$

$$= -\frac{\pi}{4}(10-w)^2(-1)\frac{dw}{dt}$$

$$= \frac{\pi}{4}(10-w)^2\frac{dw}{dt}$$

$$\Rightarrow \frac{dw}{dt} = \frac{4}{\pi}\frac{1}{(10-w)^2}\frac{dV_{water}}{dt}$$

We have $w = 7.5$ m and $\frac{dV_{water}}{dt} = 3.14\ m^3/min \approx \pi\ m^3/min$. Hence;

$$\frac{dw}{dt} = \frac{4}{\pi}\frac{1}{(10-7.5)^2}\pi$$

$$= \frac{4}{(2.5)^2}$$

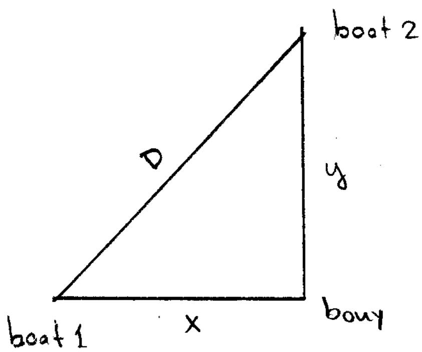
$$= 0.64\ m/min$$

Example

A boat passes a fixed buey at 9 am heading due west at 3 mph. Another boat passes the same buoy at 10 am heading due north at 5 mph. How fast is the distance between the boats changing at 11:30 am ?

Solution:

At 11:30 am the boat heading west will be a distance (3 mph)(11:30 − 9 hr) from the bouy, i.e., $^{15}/_{2}$ miles west. The boat heading north will be (5 mph)(11:30 − 10 hr) from the bouy, i.e. $^{15}/_{2}$ miles north.



We want to know $\frac{dD}{dt}$ when the boats are at these positions. So, if x is the distance from the bouy to the first boat (the one heading west) and y is the distance from the bouy to the second boat,

$$x^2 + y^2 = D^2$$

$$\Rightarrow \frac{d}{dt}(x^2 + y^2) = \frac{d}{dt} D^2$$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dx} = 2D\frac{dD}{dt}$$

$$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dx} = D\frac{dD}{dt}$$

$$\Rightarrow \frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dx}\right)$$

We are given that $\frac{dx}{dt} = 3$ mph and $\frac{dy}{dt} = 5$ mph. We have computed that $x = y = 15/2$ mi at 11:30 am and $D = \sqrt{x^2+y^2} = \sqrt{2(15/2)^2} = \frac{15}{2}\sqrt{2}$ mi at that time. Hence,

$$\frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{2}{\sqrt{2}\cdot 15}\left(\frac{15}{2}\cdot 3 + \frac{15}{2}\cdot 5\right)$$

$$= \frac{1}{\sqrt{2}}(3+5)$$

$$= \frac{8}{\sqrt{2}}\ \frac{mi}{hr}$$

$$\approx 5.65\ \frac{mi}{hr}$$

Example

At a certain moment a sample of gas obeying Boyle's law $pV = \text{constant}$, occupies a volume of 1000 in^3 at a pressure p of 10 psi. If the gas is being compressed at the rate of 12 in^3/min, find the rate at which the pressure is increasing at the instant when the volume is 600 in^3.

Solution:

We have $pV = \text{constant}$

$V = 1000$ in^3 where $p = 10$ psi

$\frac{dV}{dt} = -12$ in^3/min (neg because we are compressing the gas)

What we want to know is $\frac{dp}{dt}$ when $V = 600$ in^3.

So, write

$$pV = c$$

where c is a constant. Differentiating wrt t:

$$\frac{d}{dt} pV = \frac{d}{dt} c$$

$$\Rightarrow p\frac{dV}{dt} + V\frac{dp}{dt} = 0$$

$$\Rightarrow \frac{dp}{dt} = -\frac{p}{V}\frac{dV}{dt}$$

To evaluate this when $V = 600 \text{ in}^3$ we need to know P at that volume. So go back to $pV = c$ and compute c knowing $V = 1000 \text{ in}^3$ when $p = 10$ psi (which we were given).

$$c = pV$$

$$= (10 \text{ psi})(1000 \text{ in}^3)$$

$$= 10^4 \text{ psi-in}^3$$

Then

$$p = \frac{c}{V}$$

so where $V = 600 \text{ in}^3$;

$$p = \frac{10^4 \text{ psi-in}^3}{600 \text{ in}^3}$$

$$= \frac{10^4}{600} \text{ psi}$$

$$= \frac{100}{6} \text{ psi}$$

$$= 50/3 \text{ psi}$$

Thus when $V = 600 \text{ in}^3$ and $P = {}^{50}/_3$ psi and $dV/dt = -12 \text{ in}^3/\text{min}$

$$\frac{dP}{dt} = -\frac{P}{V}\frac{dV}{dt}$$

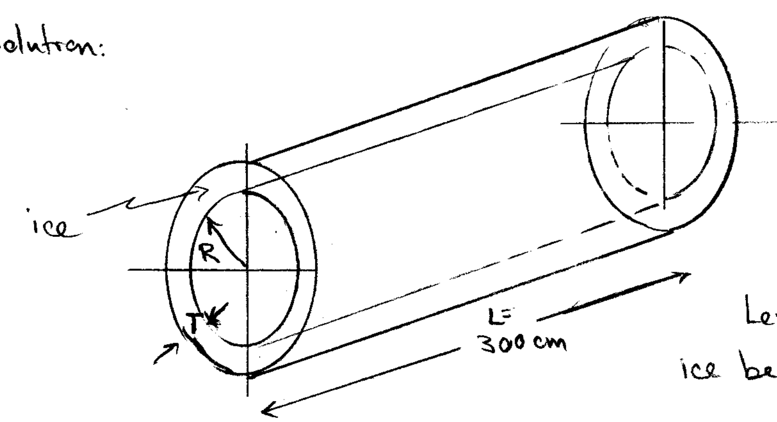
$$= -\frac{(50/3 \text{ psi})}{(600 \text{ in}^3)}\left(-12\frac{\text{in}^3}{\text{min}}\right)$$

$$= \frac{2}{3}\frac{\text{psi}}{\text{min}}$$

Example

An open pipe with length 3 m and outer radius 10 cm has an outer layer of ice that is melting at the rate of $2\pi \text{ cm}^3/\text{min}$. How fast is the thickness of the ice decreasing when the ice is 2 cm thick?

Solution:



Let the outer radius of the pipe be $R = 10$cm. The length of the pipe is $L = 3\text{m} = 300$ cm. Let the thickness of the ice be given by T.

We are given: $R = 10$cm

$L = 300$ cm

$\frac{dV}{dt} = -2\pi \frac{\text{cm}^3}{\text{min}}$ where V = volume of ice

We want $\frac{dT}{dt}$ when $T = 2$ cm

The radius of the pipe plus ice is:

$$r = R + T$$

The volume of a solid cylinder of the radius and length L is:

$$V_{outer} = \pi r^2 L$$

$$= \pi (R+T)^2 L$$

The volume of a solid cylinder of radius R (the inner radius of the ice) is:

$$V_{inner} = \pi R^2 L$$

So the volume of the ice of thickness T is:

$$V = \pi (R+T)^2 L - \pi R^2 L$$

$$\Rightarrow \frac{d}{dt} V = \frac{d}{dt} \left[\pi (R+T)^2 L - \pi R^2 L \right]$$

$$\Rightarrow \frac{dV}{dt} = \pi L \, 2(R+T) \frac{d}{dt} (R+T$$ since R and L are constants

$$\Rightarrow \frac{dV}{dt} = 2\pi L (R+T) \frac{dT}{dt}$$ since R is a constant

Thus;

$$\frac{dT}{dt} = \frac{1}{2\pi L(R+T)} \frac{dV}{dt}$$

Plugging in values:

$$\frac{dT}{dt} = \frac{1}{2\pi(300)(10+2)} \left(-2\pi \frac{cm^3}{min}\right)$$

$$= -\frac{1}{3600} \frac{cm}{min}$$

$$\approx 4 \frac{mm}{day}$$