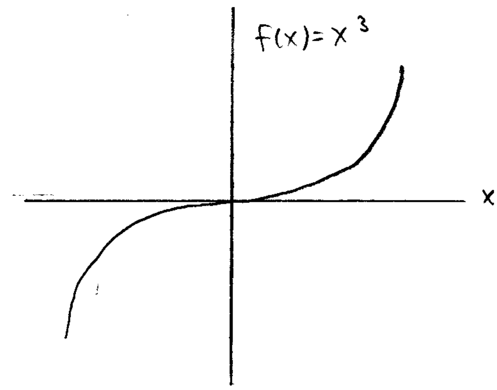
APPM 1350: Sections 3.3 and 3.4: 1st Deriv. Test for Local Extrema & Graphing

An Important Point

Recall that a critical point of f in the interior of its domain is a point where either $f'(x)=0$ or $f'(x)$ is undefined. If $f(x)$ has local extrema in that open interval, they extrema will occur at critical points. **However**, not all critical points correspond to extrema. For example:

$$f(x)=x^3$$

$$\Rightarrow f'(x)=3x^2$$



Note that $f'(0)=0$, so $x=0$ is a critical point but it is not an extrema.

What this says is the the critical points are candidates for extrema. Or put differently, if an extrema occurs in the interior of a domain, it will occur at a critical point. But not all critical points are extrema. Being a critical point is a necessary but not sufficient condition for being a point where there is an extrema.

The First Derivative Test for Local Extrema

Local extreme values will occur at c in the interior of D when

a) c is a critical point
b) $f'(x)$ changes from positive to negative at $x=c$ (local max)
or
$f'(x)$ changes from negative to positive at $x=c$ (local min)

On $[a,b]$, $f(a)$ is an extrema if

$f'(x) < 0$ for $x \in (a, a+\epsilon)$ (local max)

or

$f'(x) > 0$ for $x \in (a, a+\epsilon)$ (local min

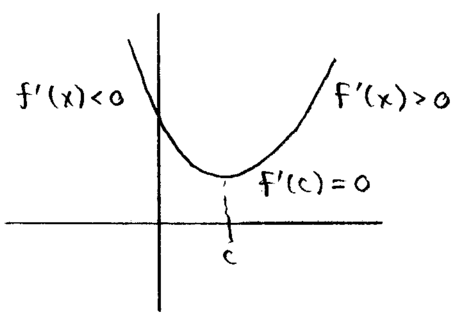
$f(b)$ is an extrema if

$f'(x) < 0$ for $x \in (b-\epsilon, b)$ (local min)

or

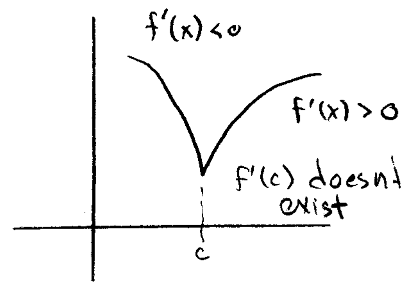
$f'(x) > 0$ for $x \in (b-\epsilon, b)$ (local max)

Examples

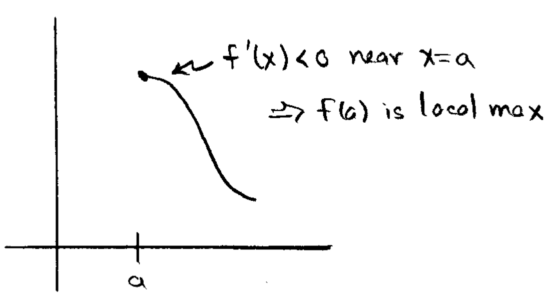


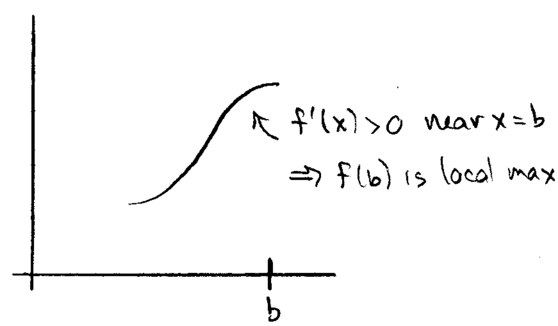
$f(c)$ is a local minimum

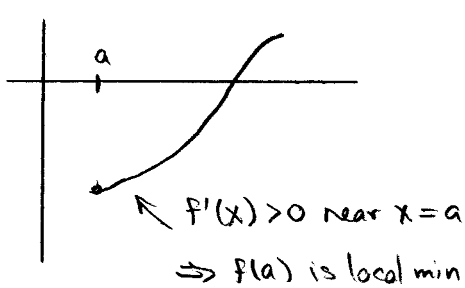
$f'(c)=0$

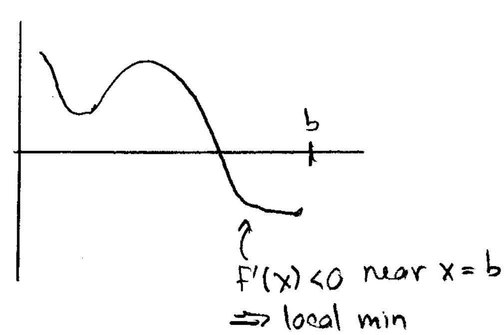


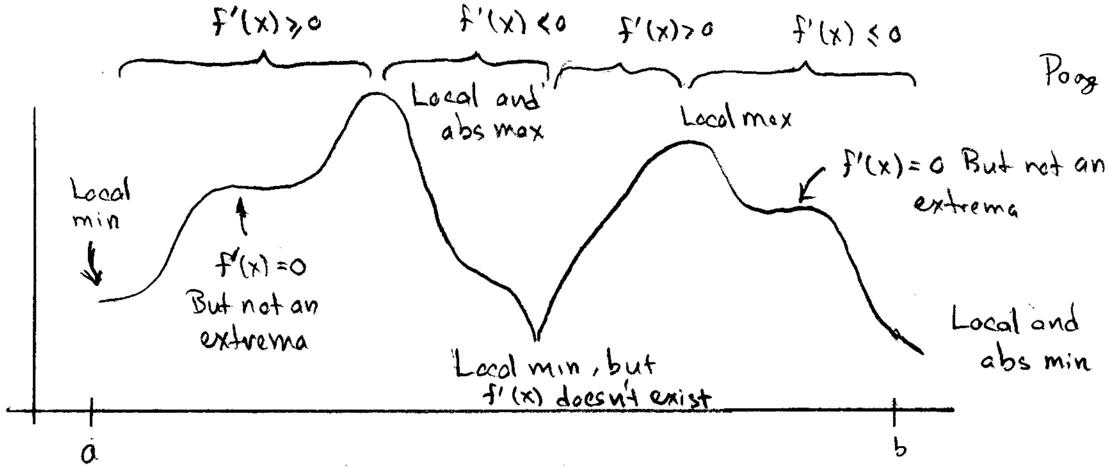
$f(c)$ is a local minimum but $f'(c)$ doesn't exist.









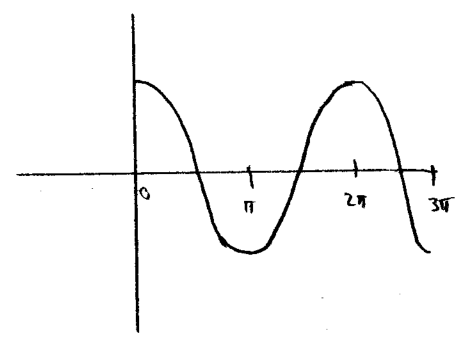


Example

Find the extrema of $f(x) = \cos x$ on $[0, 3\pi]$

Soln:
$$f(x) = \cos x$$
$$f'(x) = -\sin x$$

The critical points are where $f'(x) = 0$, hence



$$-\sin x = 0$$
$$\Rightarrow x = 0, \pi, 2\pi, 3\pi$$

Note that

$$f'(x) = \begin{cases} \text{negative} & x \in [0, \pi] \\ \text{positive} & x \in [\pi, 2\pi] \\ \text{negative} & x \in [2\pi, 3\pi] \end{cases}$$

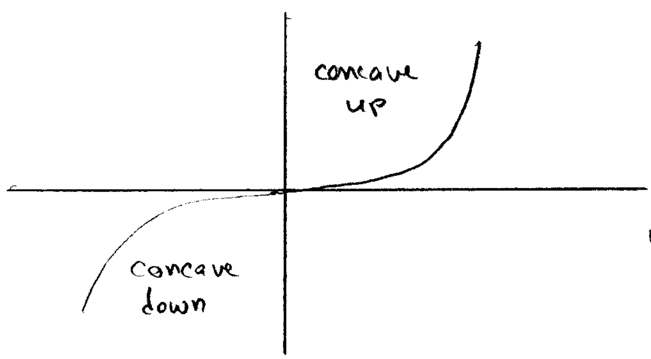
On $(0, \epsilon)$ we have $f'(x) < 0$ so $f(0) = 1$ is a maximum. Next, at $x = \pi$ the derivative goes from neg to pos which means $f(\pi) = -1$ is a minimum. At $x = 2\pi$ $f'(x)$ goes from positive to negative, so $f(2\pi) = 1$ is a maximum. Finally, on $x \in (3\pi - \epsilon, 3\pi)$ $f'(x)$ is negative, hence $f(3\pi) = -1$ is a minimum. All are local extrema and technically any one of the minima and any one of the maxima are absolute extrema as well.

Concavity

Def The graph of a differentiable function $y=f(x)$ is <u>concave up</u> on an interval where y' is increasing and <u>concave down</u> on an interval where y' is decreasing.

Example

Consider $y=x^3$. Then $y'=3x^2$. This is decreasing for $x<0$ and increasing on $x>0$. Hence the curve $y=x^3$ is concave down for $x<0$ and concave up for $x>0$.



Concave down means the graph "opens downward" while concave up means it "opens upward".

The Second Derivative Test for Concavity Let $y=f(x)$ be twice differentiable on an interval I.

a) If $y''>0$ on I then the graph of f over I is concave up.
b) If $y''<0$ on I then the graph of f over I is concave down.

Using the same example as above;

$$y=x^3$$
$$y'=3x^2$$
$$y''=6x$$

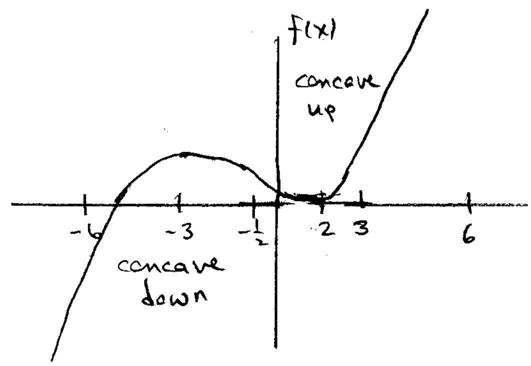
Clearly $y''(x)<0$ for $x<0 \Rightarrow$ concave down and $y''(x)>0$ for $x>0 \Rightarrow$ concave up. At $x=0$ the concavity changes. This is called a <u>point of inflection</u>.

Def A point where the graph of a function has a tangent line and where concavity changes is called a <u>point of inflection</u>.

If $y=f(x)$ is twice differentiable, then $y''=0$ at a point of inflection. However, just because $y''=0$ does not guarantee that is an inflection point (necessary but not sufficient)

<u>Example</u>

$$f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$$



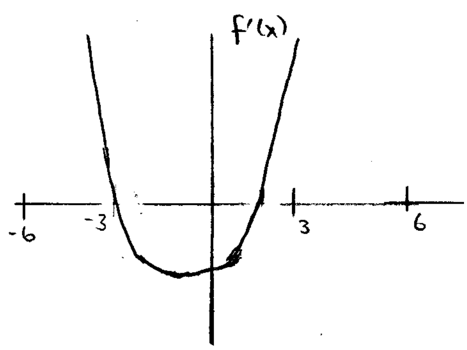
$$\Rightarrow f'(x) = x^2 + x - 6$$

$$f''(x) = 2x + 1$$

Critical points: $f'(x)=0 \Rightarrow x^2+x-6=0$

$\Rightarrow (x+3)(x-2)=0$

$\Rightarrow x=-3$ and $x=2$



Concavity: $f''(x)=0 \Rightarrow 2x+1=0$

$\Rightarrow x=-\frac{1}{2}$

Then: $f(-3)=\frac{43}{2}$ is a local maximum, $f(-\frac{1}{2})$ is a point of inflection, and $f(2)=\frac{2}{3}$ is a local minimum. The curve is concave down for $x<-\frac{1}{2}$ and concave up for $x>-\frac{1}{2}$.

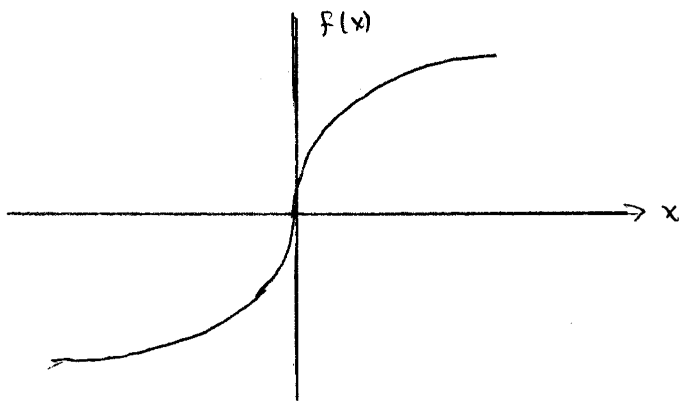
Example

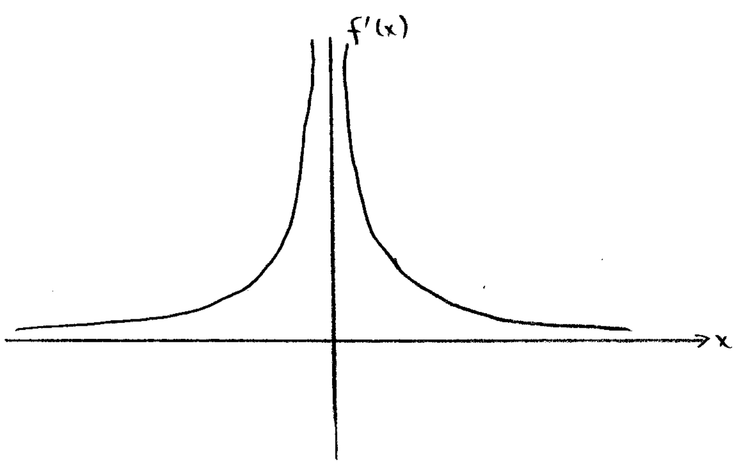
Consider $f(x) = x^{1/3}$. Then

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$$

$$f''(x) = -\frac{2}{9} x^{-5/3}$$

$f'(x) \neq 0$ for $x \neq 0$ and undefined at $x = 0$. $f''(x) > 0$ for $x < 0$ (hence concave up) and $f''(x) < 0$ for $x > 0$ (hence concave down). $x = 0$ is a point of inflection, but note that $f''(x)$ is undefined there.





Example

Consider $f(x) = x^4$. Then:

$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$

Clearly $f(0) = f'(0) = f''(0)$. Also, $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$, so $f(0) = 0$ is a local and absolute minimum. As for concavity, $f''(x) \geq 0$ for all x. So the curve is convex up everywhere. At $x = 0$ there is no inflection point since $f''(x)$ does not change sign.

The Second Derivative Test for Local Extremes

> If $f'(c) = 0$ and $f''(c) < 0$ then f has a local max at $x = c$
> If $f'(c) = 0$ and $f''(c) > 0$ then f has a local min at $x = c$

Example

Find and classify the extrema of $g(x) = 2x^3 - 9x^2 + 36$

Solution:
$$g'(x) = 6x^2 - 18x$$
$$g''(x) = 12x - 18$$

The critical points are where $g'(x) = 0 \Rightarrow 6x^2 - 18x = 0$
$\Rightarrow 6x(x-3) = 0$
$\Rightarrow x = 0$ and $x = 3$

Next; $g''(0) = -18 < 0$ and $g''(3) = 18 > 0$. So $g(0) = 36$ is a local maximum and $g(3) = 9$ is a local minimum.

Graphing Curves

A strategy for graphing $y = f(x)$

1. Determine the domain of the function.
2. Calculate y' and, if convenient, y''
3. Set $y' = 0$ to find any critical points. Determine whether they are min or max
4. Use y' to determine the intervals on which y is either increasing or decreasing.
5. Use y'' to determine if the curve is concave up or down.
6. Look for vertical asymptotes where the function is undefined.
7. Look for horizontal asymptotes.
8. Find the x and y intercepts
9. Locate any corner points (i.e., where y' is discontinuous)
10. Indicate any cusps ($y' \to +\infty$ from one side and $-\infty$ from the other)

Example

Let's graph the function from the previous example;

$$g(x) = 2x^3 - 9x^2 + 36$$

(Step 1) The domain is $D = \mathbb{R}$. Further:

(Step 2)

$$g'(x) = 6x^2 - 18x$$
$$= 6x(x-3)$$
$$g''(x) = 12x - 18$$
$$= 6(2x-3)$$

$g'(x) = 0 \Rightarrow x = 0$ and $x = 3$

(step 3)

$g''(0) = -18 < 0 \Rightarrow g(0) = 36$ is a local max

$g''(3) = 18 > 0 \Rightarrow g(3) = 9$ is a local min

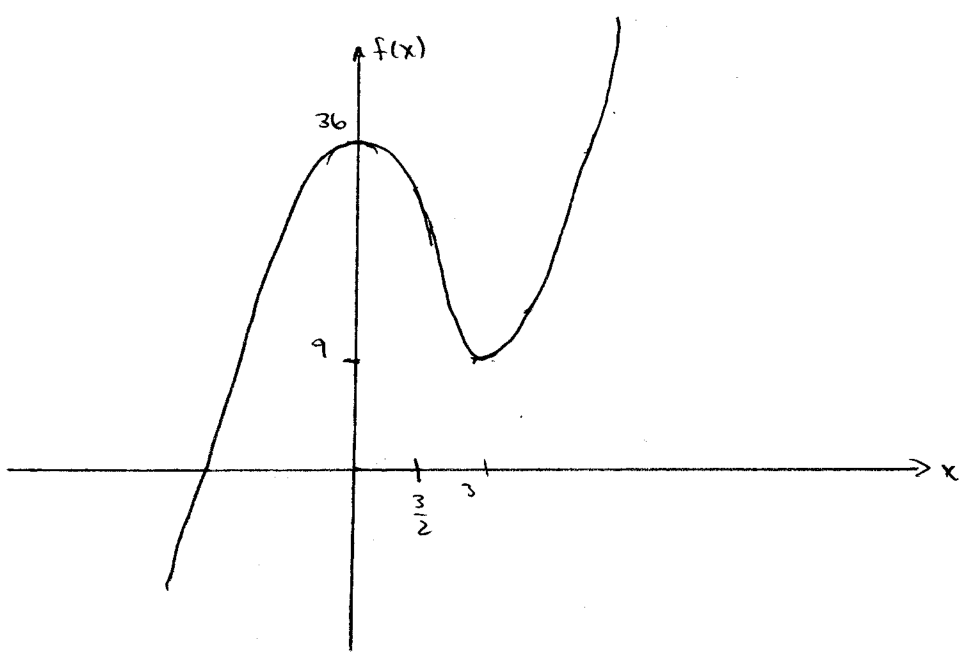
(step 4)

$g'(x) > 0$ for $x < 0$ (increasing)
$g'(x) < 0$ for $0 < x < 3$ (decreasing)
$g'(x) > 0$ for $x > 3$ (increasing)

(step 5)

$g''(x) < 0$ for $x < \frac{3}{2}$ (concave down)
$g''(x) > 0$ for $x > \frac{3}{2}$ (concave up)

There are no vertical or horizontal asymptotes.



Example
(# 42, p. 218)

Sketch the general shape of f(x) given

$$f'(x) = x^2 - x - 6$$

Solution:

$f'(x) = 0 \Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0$

$\Rightarrow x = 3$ and $x = -2$ (critical points)

$f''(x) = 2x - 1 \Rightarrow f''(-2) = -4 - 1 = -5 < 0$ (local max at $x = -2$)

$f''(3) = 6 - 1 = 5 > 0$ (local min at $x = 3$)

Further; $f'(x) > 0$ for $x < -2$ (increasing)

$f'(x) < 0$ for $-2 < x < 3$ (decreasing)

$f'(x) > 0$ for $x > 3$ (increasing)

and

$f''(x) = 0$ at $x = \frac{1}{2}$

$f''(x) < 0$ for $x < \frac{1}{2}$ (concave down)

$f''(x) > 0$ for $x > \frac{1}{2}$ (concave up)