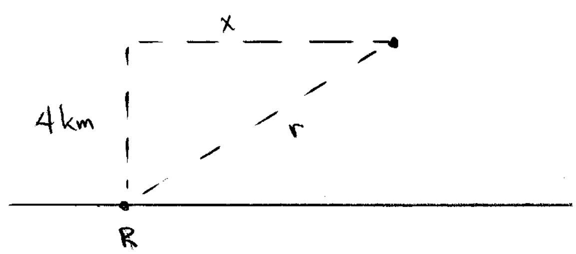
APPM 1350: Section 3.6 : Optimization

Review Problem

A plane flying parallel to the ground at a height of 4 km passes over a radar station R (located on the ground). A short time later, the radar equipment reveals that the distance between the plane and the station is 5 km and that the distance between the plane and the station is increasing at a rate of 300 km/hr. At that moment, how fast is the plane moving horizontally?

Solution.



Based on the picture above, when $r = 5$ km then $\frac{dr}{dt} = 300$ km/hr. What we want to find is $\frac{dx}{dt}$. Now,

$$(4\text{ km})^2 + x^2 = r^2$$

$$\Rightarrow 16 + x^2 = r^2 \qquad \text{(all units in km)}$$

$$\Rightarrow \frac{d}{dt}(16 + x^2) = \frac{d}{dt} r^2$$

$$\Rightarrow 2x\frac{dx}{dt} = 2r\frac{dr}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{r}{x}\frac{dr}{dt}$$

$$= \frac{r}{\sqrt{r^2 - 16}}\frac{dr}{dt}$$

Then plugging in $r = 5$ and $\frac{dr}{dt} = 300$:

$$\frac{dx}{dt} = \frac{5}{\sqrt{25-16}} \, 300 \, \frac{km}{hr}$$

$$= \frac{5}{\sqrt{9}} \, 300 \, \frac{km}{hr}$$

$$= 500 \, \frac{km}{hr}$$

Applied Min-Max Problems

We now turn our attention to the problem of optimization. This means minimizing or maximizing some part of a problem, or more generally maximizing one thing while minimizing something else (for example maximizing the price of a commodity while minimizing its manufacturing cost). Such problems are also called min-max problems.

Like related rates problems, min-max problems are typically word problems and are probably best taught by example. So this lecture will consist of examples drawn from real-world type problems. But first some general hints on strategy for solving min-max problems

Strategy for Solving Min-Max Problems

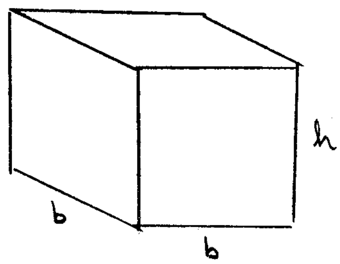
1. Read the problem (duh! I only put this in because the book did)
2. Draw a picture
3. Introduce variables.
4. Identify the unknown
5. Test the critical points and end points to identify the minima and/or maxima of interest.

Okay, enough theory. Let's now do bunches of examples.

Example

A closed box with a square base is to contain 252 ft^3. The bottom costs \$5 per ft^2, the top costs \$2 per ft^2, and the sides cost \$3 per ft^2. Find the dimensions that will minimize the cost.

Solution



Let the length the each side of the base be equal to b and the height of the box be h. The area of the base and top is b^2 and the area of each side is bh. The volume of the box is:

$$V = b^2 h \qquad (1)$$

and the cost of building it is:

$$C = \underbrace{5b^2}_{\text{bottom}} + \underbrace{2b^2}_{\text{top}} + 4\underbrace{(3bh)}_{\text{each side}}$$

$$= 7b^2 + 12bh \qquad (2)$$

We want $V = 252$ ft^3 and the minimum cost. Now, equation (2) for the cost has independent variables b and h. But these are related by the volume in equation (1). Hence

$$V = b^2 h$$

$$\Rightarrow h = \frac{V}{b^2}$$

Hence:

$$C = 7b^2 + 12bh$$

$$= 7b^2 + 12b\left(\frac{V}{b^2}\right)$$

$$= 7b^2 + 12\frac{V}{b}$$

$$= 7b^2 + 12\frac{252}{b}$$

$$= 7b^2 + \frac{3024}{b}$$

Clearly we are only interested in $b > 0$. For that domain $C(b)$ is differentiable, and the point where $C(b)$ is smallest (minimum) will be where $C'(b) = 0$. So:

$$C'(b) = \frac{d}{db}(7b^2 + 3024b^{-1})$$

$$= 14b - \frac{3024}{b^2}$$

Set $C'(b) = 0$ and solve for b:

$$0 = 14b - \frac{3024}{b^2}$$

$$\Rightarrow 14b = \frac{3024}{b^2}$$

$$\Rightarrow b^3 = \frac{3024}{14}$$

$$= 216$$

$$\Rightarrow b = 6 \text{ ft}$$

Let's make sure this really is a minimum and not a maximum. Use the 2nd derivative test:

$$C''(b) = \frac{d}{db}\left(14b - 3024\,b^{-2}\right)$$

$$= 14 + 2(3024)\,b^{-3}$$

$$> 0 \quad \text{for } b = 6$$

Since the 2nd derivative is positive at this critical point, this is indeed a minimum. So we have the length of the base. What about the height h? From (1)

$$V = b^2 h$$

$$\Rightarrow h = \frac{V}{b^2}$$

$$= \frac{252}{6^2}$$

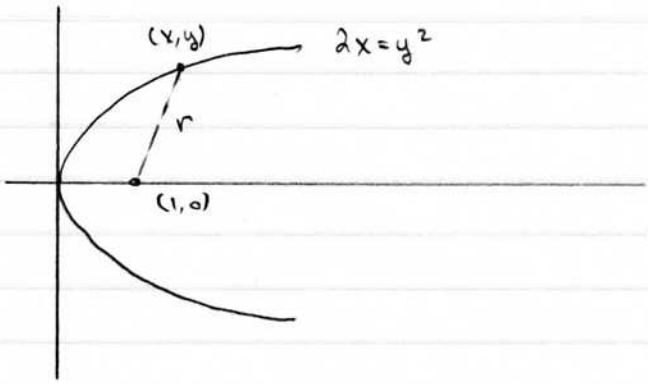
$$= 7 \text{ ft}$$

So the least expensive box of volume $252\,ft^3$ will be base 6 ft and height 7 ft. The cost is \$756.

Example

Find the point(s) on the parabola $2x = y^2$ closest to the point $(1,0)$.

Solution



The distance r is given by

$$r^2 = (x-1)^2 + (y-0)^2$$

$$= (x-1)^2 + y^2$$

From the equation for the parabola, $y^2 = 2x$. Hence

$$r^2 = (x-1)^2 + 2x$$

We are interested in x such that r is a minimum. The function is differentiable for all $x \geq 0$. Hence look for where $dr/dx = 0$ for a minimum:

$$\frac{d}{dx} r^2 = \frac{d}{dx}\left[(x-1)^2 + 2x\right]$$

$$\Rightarrow 2r\frac{dr}{dx} = 2(x-1) + 2$$

$$\Rightarrow r\frac{dr}{dx} = x$$

$$\Rightarrow \frac{dr}{dx} = \frac{x}{r}$$

Setting $dr/dx = 0$, then

$$0 = \frac{x}{r}$$

$$\Rightarrow x = 0$$

since clearly $r > 0$ for all x (look at the picture). So $x = 0$ is a critical point. One can tell by looking at the picture that this is a minimum. However, if you want to be sure, then use the 2nd derivative test:

$$\frac{dr}{dx} = \frac{x}{r}$$

$$= \frac{x}{\sqrt{(x-1)^2 + 2x}}$$

$$= \frac{x}{\sqrt{x^2+1}}$$

$$\Rightarrow \frac{d^2r}{dx^2} = \frac{d}{dx}\frac{x}{\sqrt{x^2+1}}$$

$$= \frac{(x^2+1)^{1/2} - x\frac{1}{2}(x^2+1)^{-1/2}}{x^2+1}$$

$$= \frac{(x^2+1) - \frac{1}{2}x}{(x^2+1)^{3/2}}$$

This is equal to one at $x=0$, Since it is positive then $x=0$ is a minimum point. Note that at $x=0$ then $y=0$, so the point on the parabola closest to $(1,0)$ is the point $(0,0)$.

Example

Suppose the reaction $R(x)$ to a dose x of a drug is given by

$$R(x) = Ax^2(B-x)$$

where $A>0$ and $B>0$. The sensitivity $S(x)$ of the body to a dose of size x is defined to be $R'(x)$. Assume a negative reaction is a bad thing.

a) What is the domain of R? What seems to be the physical meaning of A and B?

b) For what value of x is R a maximum?

c) For what value of x is the sensitivity a maximum?

d) Why is it called sensitivity?

Solution:

a) The domain of $R(x)$ is $\mathbb{R}$. However, there is no such thing as a negative dose, so clearly $x \geq 0$. We also don't want negative reactions and that means $x \leq B$. Thus we define the domain to be $\{x \in \mathbb{R} : 0 \leq x \leq B\}$.

Biologically, B represents the largest permissible dose (anything larger will provoke a negative reaction). The constant A controls how quickly the reaction occurs.

b) $R(x)$ is differentiable everywhere on its domain. So look for critical points in the interior and also look for the values at the end points. For critical points we want $R'(x) = S(x) = 0$. So

$$R'(x) = \frac{d}{dx} Ax^2(B-x)$$

$$= \frac{d}{dx} ABx^2 - Ax^3$$

$$= 2ABx - 3Ax^2$$

$$= Ax(2B-3x)$$

$R'(x) = 0$ at $x = \frac{2B}{3}$ in the interior of the domain. Next;

$$R''(x) = \frac{d}{dx}\left(2ABx - 3Ax^2\right)$$

$$= 2AB - 6Ax$$

$$= 2A(B - 3x)$$

At $x = \frac{2B}{3}$, $R''\left(\frac{2B}{3}\right) = 2A(B - 2B) < 0$. So $x = \frac{2B}{3}$ is a maximum, But is it an absolute minimum? At the endpoints;

$$R(0) = 0$$
$$R(B) = 0$$

At at $x = \frac{2B}{3}$,

$$R\left(\frac{2B}{3}\right) = A\left(\frac{2B}{3}\right)\left(B - \frac{2B}{3}\right)$$

$$= \frac{2AB^2}{9}$$

So yes, $x = \frac{2B}{3}$ is the absolute maximum on $x \in [0, B]$.

c) We have from above:

$$S(x) = R'(x) = Ax(2B - 3x)$$

We also have:

$$S'(x) = R''(x) = 2A(B - 3x)$$

from which we get

$$S''(x) = -6A$$

Find the critical points:

$$S'(x) = 0$$

$$\Rightarrow \quad 2A(B-3x) = 0$$

$$\Rightarrow \quad x = \frac{B}{3}$$

Then note that $S''\left(\frac{B}{3}\right) < 0$ (in fact $S''(x) < 0 \ \forall \ x \in [0, B]$). So $x = B/3$ is maximum. But before saying it is the absolute maximum, check the values at the endpoints:

$$S(0) = 0$$

$$S(B) = AB(2B - 3B) = -AB^2 < 0$$

Since $S(B/3) = \frac{AB^2}{3} > 0$ that is indeed the absolute maximum.

d) It is called sensitivity because it represents the rate of change of the reaction. The greater $S(x)$ then the faster $R(x)$ is changing with dose x, hence the body is more sensitive (reactive).

Example

The range R of a projectile whose muzzle velocity in meters per second is v, and whose angle of elevation in radians is θ, is given by

$$R = \frac{v^2 \sin 2\theta}{g}$$

where g is the acceleration of gravity. Which angle of elevation gives the maximum range of the projectile?

Solution:

The domain is $\theta \in [0, \frac{\pi}{2}]$ by examination of the picture. Now;

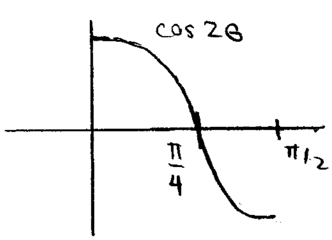
$$\frac{dR}{d\theta} = \frac{d}{d\theta} \frac{v^2 \sin 2\theta}{g}$$

$$= \frac{v^2}{g} \frac{d}{d\theta} \sin 2\theta$$

$$= \frac{v^2}{g} \cos 2\theta \frac{d}{d\theta} 2\theta$$

$$= \frac{2v^2}{g} \cos 2\theta$$

Now;



$$\frac{dR}{d\theta} = 0 \Rightarrow \frac{2v^2}{g} \cos 2\theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

This is a maximum by inspection. But just to be sure;

$$\frac{d^2R}{d\theta^2} = -\frac{4V^2}{g}\sin 2\theta$$

This equals $-\frac{4V^2}{g} < 0$ at $\theta = \pi/4$. So indeed, $\theta = \pi/4$ is a local maximum.
Also, at the endpoints:

$$R(0) = 0$$
$$R(\pi/2) = 0$$

Since $R(\pi/4) = V^2/g$, then indeed $\theta = \pi/4$ is the absolute maximum.

<u>Example</u>

A truck driving over a flat interstate at a constant speed of 50 mph gets 4 mile to the gallom (mpg). Fuel costs \$2.50 per gallon. For each mile per hour increase in speed the truck loses a tenth of a mile per gallon in its mileage. Drivers get \$40.00 per hour in wages, and fixed costs for running the truck amount to \$25 per hour. What constant speed (between 50 mph and the speed limit of 70 mph) should a dispatcher require on a straight run through 260 miles of kansas interstate to minimize the total cost of operating the truck?

Solution:

Distance s is related to velocity and time as:

$$s = vt$$

when the velocity (speed) is constant. In our case, $s = 260$ miles.

So

$$260 = vt$$

The mileage (mpg) of the truck as a function of velocity is:

$$mpg = 4 - \frac{1}{10}(v-50)$$

The cost of operating the truck as a function of time <u>and</u> velocity is:

$$C(v,t) = \underbrace{(2.50\ \$/gal)\frac{260(mi)}{mpg\left(\frac{mi}{gal}\right)}}_{\text{fuel costs}} + \underbrace{(40\ \$/hr)t}_{\text{wages}} + \underbrace{(25\ \$/hr)t}_{\text{operating costs}}$$

$$= \frac{(2.5)(260)}{4-\frac{1}{10}(v-50)} + 65t$$

$$= \frac{6500}{40 - v + 50} + 65t$$

$$= \frac{6500}{90-v} + 65t$$

Since $260 = vt$, then $t = 260/v$, hence

$$C(v) = \frac{6500}{90-v} + \frac{65(260)}{v}$$

$$= \frac{6500}{90-v} + \frac{6500}{v}$$

$$= 6500\left(\frac{1}{90-v} + \frac{1}{v}\right)$$

Then;

$$C'(V) = 6500 \frac{d}{dV}\left[(90-V)^{-1} + V^{-1}\right]$$

$$= 6500\left[-(90-V)^{-2}(-1) - V^{-2}\right]$$

$$= 6500\left[\frac{1}{(90-V)^2} - \frac{1}{V^2}\right]$$

$$= 6500 \frac{V^2 - (90-V)^2}{V^2(90-V)^2}$$

For $0 < V < 90$ the denominator is nonzero. So $C'(V) = 0$ means the numerator is zero. Hence

$$V^2 - (90-V)^2 = 0$$

$$\Rightarrow \cancel{V^2} - (90^2 - 2(90)V + \cancel{V^2}) = 0$$

$$\Rightarrow (90)^2 - 2(90)V = 0$$

$$\Rightarrow V = \frac{90}{2} \text{ mph}$$

$$= 45 \text{ mph}$$

This is indeed a minimum, but it is outside the range of speeds of interest, i.e., between 50 and 70 mph. So within the range of interest there are no critical points. The extrema will then be at the endpoints. So check:

$$C(50) = \$292.50$$
$$C(70) = \$417.86$$

Thus $V = 50$ mph is the absolute minimum. The driver should drive at this speed to minimize his costs if his permissible speeds are between 50 mph and 70 mph.

Example

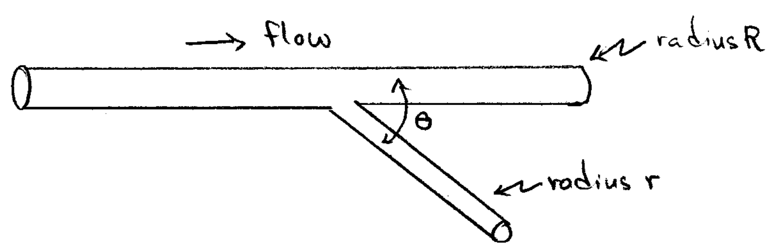
According to Poiseuille's law, if blood flows into a straight blood vessel of radius r branching off another straight blood vessel of radius R at an angle θ, the total resistance T of the blood in the branching vessel is given by

$$T = C\left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right)$$

where a, b, and C are constants and $r < R$. Show that the total resistance is minimized when

$$\cos\theta = \left(\frac{r}{R}\right)^4$$

Solution:



From the equation for T and our picture the domain of $T(\theta)$ is $\theta \in (0,\pi)$. In this domain $T(\theta)$ is differentiable. Then:

$$T'(\theta) = \frac{d}{d\theta} C\left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right)$$

Recall: $\frac{d}{d\theta}\cot\theta = -\csc^2\theta$

$\frac{d}{d\theta}\csc\theta = -\csc\theta\cot\theta$

$$= \frac{C}{R^4}\frac{d}{d\theta}(a - b\cot\theta) + \frac{Cb}{r^4}\frac{d}{d\theta}\csc\theta$$

$$= -\frac{bC}{R^4}(-\csc^2\theta) + \frac{Cb}{r^4}(-\csc\theta\cot\theta)$$

$$= bC\csc\theta\left[\frac{1}{R^4}\csc\theta - \frac{1}{r^4}\cot\theta\right]$$

Since $\csc\theta \neq 0$ for $\theta \in (0,\pi)$, then $T'(\theta) = 0$ means

$$\frac{1}{R^4}\csc\theta - \frac{1}{r^4}\cot\theta = 0$$

$$\Rightarrow \frac{\csc\theta}{R^4} = \frac{\cot\theta}{r^4}$$

$$\Rightarrow (\sin\theta)(\cot\theta) = \left(\frac{r}{R}\right)^4$$

$$\Rightarrow \cos\theta = \left(\frac{r}{R}\right)^4$$

as expected.

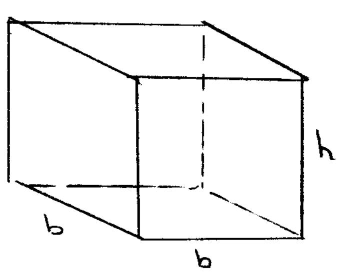
Example

A piece of wire 100 cm long is to be cut into several pieces and used to construct the skeleton of a box with a square base.

a) What is the largest possible volume that such a box can have?

b) What is the largest possible surface area?

Solution:



Irrespective of volume or surface area, the length of wire required to build the skeleton of the box is:

$$L = 8b + 4h$$

We have 100 cm of wire, hence

$$100 = 8b + 4h$$

a) The volume of the box is:

$$V = b^2 h$$

Using $100 = 8b + 4h \Rightarrow h = 25 - 2b$, we have

$$V(b) = b^2(25-2b)$$
$$= 25b^2 - 2b^3$$

Note that we are constrained by $0 < b < \frac{25}{2}$ (let $h=0$ in $100 = 8b+4h$).
Now,

$$V'(b) = 50b - 6b^2$$

then

$$V'(b) = 0 \Rightarrow 50b = 6b^2$$
$$\Rightarrow 6b = 50$$
$$\Rightarrow b = \frac{50}{6}$$
$$= \frac{25}{3}$$

Is this a maximum?

$$V''(b) = 50 - 12b$$

$$\Rightarrow V''\left(\frac{25}{3}\right) = 50 - 12\left(\frac{25}{3}\right)$$

$$= -50$$

$$< 0$$

Yes, it is a maximum. There are no endpoints to check because the domain $(0 < b < \frac{25}{2})$ is open. So the largest volume such a box can have is:

$$V\left(\frac{25}{3}\right) = \left(\frac{25}{3}\right)^2\left(25 - 2\cdot\frac{25}{3}\right) = \left(\frac{25}{3}\right)^3 \text{ cm}^3.$$

b) The surface area is:

$$A = \underbrace{2b^2}_{\text{top \& bottom}} + \underbrace{4bh}_{\text{sides}}$$

(assuming the box is closed). Then using $h = 25 - 2b$ again;

$$\begin{aligned} A &= 2b^2 + 4b(25-2b) \\ &= 2b^2 + 100b - 8b^2 \\ &= 100b - 6b^2 \\ &= 2b(50-3b) \end{aligned}$$

$$\begin{aligned} A'(b) &= 100 - 12b \\ &= 4(25-3b) \end{aligned}$$

$$A''(b) = -12$$

Then $A'(b) = 0 \Rightarrow b = \frac{25}{3}$ (again!!!). Since $A''(b) < 0 \ \forall b$, this is a maximum. So the maximum area of such a box is:

$$\begin{aligned} A\left(\tfrac{25}{3}\right) &= 2\left(\tfrac{25}{3}\right)\left(50 - 3\tfrac{25}{3}\right) \\ &= 2\left(\tfrac{25}{3}\right)(25) \\ &= \frac{1250}{3} \text{ cm}^2 \end{aligned}$$

Enough problems for one night!