APPM 1350: Section 4.3 : Integration by Substitution

A Review of Differentials

Given a function $f(x)$, then

$$df(x) = f'(x)\,dx$$

So consider a function such as $\sin x$. From above we get

$$d(\sin x) = \cos x\,dx$$

The rules for differentials are the same as for derivatives, as summarized below:

$$dc = 0 \qquad c = \text{constant}$$
$$d(cu) = c\,du$$
$$d(u+v) = du + dv$$
$$d(uv) = u\,dv + v\,du$$
$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$
$$d(u^n) = nu^{n-1}\,du$$
$$d(\sin u) = \cos u\,du$$
$$d(\cos u) = -\sin u\,du$$
$$d(\tan u) = \sec^2 u\,du$$
$$d(\cot u) = -\csc^2 u\,du$$
$$d(\sec u) = \sec u \tan u\,du$$
$$d(\csc u) = -\csc u \cot u\,du$$

(see p. 251 for more discussion of this topic).

The Substitution Method

Let's start with an example. Consider

$$\int (3x+2)^5 \, dx$$

We know how to integrate

$$\int u^5 \, du = \frac{u^6}{6} + c$$

But what about a composite function like $(3x+2)^5$? The trick is to make a substitution that changes $(3x+2)^5$ into a term we do know how to integrate, in particular u^5. So try

$$u = 3x + 2$$

Take the <u>differential of both sides</u>:

$$du = 3\,dx$$

$$\Rightarrow dx = \frac{du}{3}$$

Then substituting u for $3x+5$ and $\frac{du}{3}$ for dx we have:

$$\int (3x+2)^5 dx = \int u^5 \frac{du}{3}$$

$$= \frac{1}{3} \int u^5 \, du$$

$$= \frac{1}{3} \frac{u^6}{6} + c$$

$$= \frac{(3x+5)^6}{18} + c$$ Resubstituting $3x+5$ for u.

This is an example of the <u>substitution method</u> for integration.

Theoretically, what we are doing with substitution is using the chain rule in reverse. So consider a function $F(u)$ and another function $\phi(x)$. Then form a composite function from $F(u)$ by letting $u = \phi(x)$. Call the new function $G(x)$. So we <u>define</u> $G(x)$ as

$$F(u) = F(\phi(x)) = G(x) \tag{1}$$

By the chain rule

$$\frac{dG}{dx} = \frac{dF}{du}\frac{du}{dx}$$

$$= \frac{dF}{du}\phi'(x) \qquad \text{since } u = \phi(x) \tag{2}$$

Now define

$$f(u) = F'(u) \tag{3}$$

and

$$g(x) = G'(x) \tag{4}$$

Equivalent definitions are that

$$F(u) = \int f(u)\,du \tag{5}$$

and

$$G(x) = \int g(x)\,dx \tag{6}$$

Substituting (3) and (4) into the chain rule (2):

$$G'(x) = F'(u)\,\phi'(x)$$

$$\Rightarrow \quad g(x) = f(u)\,\phi'(x) \tag{7}$$

But from (1) we had (by definition)

$$G(x) = F(u). \tag{8}$$

which from (5) and (6) is the same as saying

$$\int g(x)\,dx = \int f(u)\,du \qquad (9)$$

Substituting (7) for $g(x)$ we get:

$$\int f(u)\,\phi'(x)\,dx = \int f(u)\,du$$

$$\boxed{\Rightarrow \int f(\phi(x))\,\phi'(x)\,dx = \int f(u)\,du \qquad \text{since } u = \phi(x)}$$

This is the general form for integrals that can be done by substitution. Look for function $f(\phi(x))$ that are multiplied by the derivative $\phi'(x)$.

<u>Example</u>

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \int (\underbrace{1+x^2}_{\phi(x)})^{-1/2}\,x\,dx$$

$$= \frac{1}{2}\int (\underbrace{1+x^2}_{\phi(x)})^{-1/2}\,\underbrace{(2x)}_{\phi'(x)}\,dx \qquad \text{Let } f(u) = u^{-1/2}$$

$$= \frac{1}{2}\int u^{-1/2}\,du$$

$$= u^{1/2} + c$$

$$= \sqrt{1+x^2} + c$$

or done just a trifle differently;

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \int x\,(1+x^2)^{-1/2}\,dx$$

$$= \int x\,u^{-1/2}\,\frac{du}{2x}$$

$$= \frac{1}{2}\int u^{-1/2}\,du$$

$$= u^{1/2} + c$$

$$= \sqrt{1+x^2} + c$$

Let $u = 1 + x^2$
$\Rightarrow du = 2x\,dx$
$\Rightarrow dx = \frac{du}{2x}$

<u>Example</u>

$$\int x \sin 2x^2\,dx = \int x \sin u\,\frac{du}{4x}$$

$$= \frac{1}{4}\int \sin u\,du$$

$$= -\frac{1}{4}\cos u + c$$

$$= -\frac{1}{4}\cos 2x^2 + c$$

Let $u = 2x^2$
$\Rightarrow du = 4x\,dx$
$\Rightarrow dx = \frac{du}{4x}$

Always look to substitute for the most complicated parts of the equation or to change a function you don't know how to integrate ($\sin 2x^2$) into one you do know how to integrate ($\sin u$).

Example

(#10, sec 4.3)

$$\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx = \int \frac{1}{\cancel{x^2}} \cos^2 u \,(-\cancel{x^2}\, du)$$

$$= -\int \cos^2 u \, du$$

$$= -\int \frac{1+\cos 2u}{2}\, du$$

$$= -\frac{1}{2}\int du - \frac{1}{2}\int \cos 2u \, du$$

$$= -\frac{1}{2}u - \frac{1}{4}\sin 2u + c$$

$$= -\frac{1}{2}\left(\frac{1}{x}\right) - \frac{1}{4}\sin\frac{2}{x} + c$$

Let $u = \frac{1}{x} = x^{-1}$

$du = -x^{-2}\,dx$

$\Rightarrow dx = -x^2\,du$

Also, use

$\cos^2 u = \dfrac{1+\cos 2u}{2}$

Example

(#22, sec 4.3)

$$\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\, dx = \int \frac{u^3}{\sqrt{x}}\, 2\sqrt{x}\, du$$

$$= 2\int u^3\, du$$

$$= 2\,\frac{u^4}{4} + c$$

$$= \frac{u^4}{2} + c$$

$$= \frac{(1+\sqrt{x})^4}{2} + c$$

Let $u = 1+\sqrt{x}$

$= 1 + x^{1/2}$

$\Rightarrow du = \frac{1}{2}x^{-1/2}\,dx$

$\Rightarrow dx = 2\sqrt{x}\,du$

Example

(#26, sec 4.3)

$$\int \tan^2 x \sec^2 x \, dx = \int u^2 \sec^2 x \frac{du}{\sec^2 x}$$

Let $u = \tan x$

$\Rightarrow du = \sec^2 x \, dx$

$\Rightarrow dx = \dfrac{du}{\sec^2 x}$

$$= \int u^2 du$$

$$= \frac{u^3}{3} + c$$

$$= \frac{\tan^3 x}{3} + c$$

Example

(#34, sec 4.3)

$$\int \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv = \int -2\, du$$

Let $u = \csc\left(\frac{v-\pi}{2}\right)$

$\Rightarrow du = -\csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)\frac{dv}{2}$

$\Rightarrow dv = -\dfrac{2\, du}{\csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)}$

$$= -2 \int du$$

$$= -2u + C$$

$$= -2 \csc\left(\frac{v-\pi}{2}\right) + c$$

This one is a bit different. Here we substituted one differential for another rather than some part of the original function and the differential as in the previous examples.

Example
(#41, sec 4.3)

$$\int \frac{1}{\theta^2} \sin\frac{1}{\theta} \cos\frac{1}{\theta}\, d\theta = \int \frac{1}{\theta^2}\, u \cos\frac{1}{\theta} \left[-\frac{\theta^2\, du}{\cos(\frac{1}{\theta})}\right]$$

$$= -\int u\, du$$

$$= -\frac{u^2}{2} + C$$

$$= -\frac{1}{2}\sin^2\left(\frac{1}{\theta}\right) + C$$

Try $u = \sin\frac{1}{\theta}$

$\Rightarrow du = \left(\cos\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right) d\theta$

$\Rightarrow d\theta = -\frac{\theta^2 du}{\cos(\frac{1}{\theta})}$

Example
(#46, sec 4.3)

$$\int \sqrt{\frac{x-1}{x^5}}\, dx = \int \frac{1}{x^2} \sqrt{\frac{x-1}{x}}\, dx$$

$$= \int \frac{1}{x^2} \sqrt{1 - \frac{1}{x}}\, dx$$

$$= \int \frac{1}{x^2} \sqrt{u}\, (x^2\, du)$$

$$= \int \sqrt{u}\, du$$

$$= \int u^{1/2}\, du$$

$$= \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3}\left(1 - \frac{1}{x}\right)^{3/2} + C$$

Try $u = 1 - \frac{1}{x}$

$\Rightarrow du = \frac{1}{x^2} dx$

$\Rightarrow dx = x^2 du$

Example

(#49, sec 4.3)

$$\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$$

Try $u = \sqrt{3(2r-1)^2+6}$

$\Rightarrow du = \dfrac{6(2r-1)}{\sqrt{3(2r-1)^2+6}}\,dr$

$$= \int \cos u \,\underbrace{\left[\frac{(2r-1)\,dr}{\sqrt{3(2r-1)^2+6}}\right]}_{=\frac{du}{6}}$$

$$= \frac{1}{6}\int \cos u\,du$$

$$= \frac{1}{6}\sin u + c$$

$$= \frac{1}{6}\sin\sqrt{3(2r-1)^2+6} + c$$

Example

$$\int \frac{6x}{2+3x}\,dx = \int 6x\,\frac{1}{u}\left(\frac{1}{3}du\right)$$

Try $u = 2+3x$

$\Rightarrow du = 3\,dx$

$\Rightarrow dx = \frac{1}{3}du$

$$= 2\int \frac{x}{u}\,du$$

But note that

$u = 2+3x$

$\Rightarrow 3x = u-2$

$\Rightarrow x = \dfrac{u-2}{3}$

Substitute this for x

$$= 2\int \left(\frac{u-2}{3}\right)\frac{du}{u}$$

$$= \frac{2}{3}\int \frac{u-2}{u}\,du$$

$$= \frac{2}{3}\int \left(1-\frac{2}{u}\right)du$$

$$= \frac{2}{3}\left(u + \frac{1}{u^2}\right) + c$$

$$= \frac{2}{3}\left(2+3x + \frac{1}{(2+3x)^2}\right) + c$$

Example

$$\int \sin^3 x \cos^4 x \, dx = \int \sin^3 x \; u^4 \left(-\frac{du}{\sin x}\right)$$

Let $u = \cos x$

$\Rightarrow du = -\sin x \, dx$

$$= -\int \sin^2 x \; u^4 \, du$$

$$= -\int (1 - \cos^2 x) u^4 \, du$$

$$= -\int (1 - u^2) u^4 \, du$$

$$= \int (u^6 - u^4) \, du$$

$$= \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$