APPM 1350: Section 4.6: Properties, Area, and the Mean Value Theorem

Review

Recall the definition of the definite integral in terms of Riemann sums:

$$\boxed{\int_a^b f(x)\,dx \equiv \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k}$$

where P is a partition $a = x_0 < x_1 < x_2, \cdots < x_n = b$, $\Delta x_k = x_k - x_{k-1}$, $c_k \in [x_{k-1}, x_k]$, and

$$\|P\| = \max_{1\le k\le n} \Delta x_k$$

From this we noted that if $f(x)$ is continuous (or piecewise continuous) on $[a,b]$, then if

$$f(x) \geq 0 \text{ for } x\in[a,b] \Rightarrow \int_a^b f(x)\,dx \geq 0$$

and if

$$f(x) \leq 0 \text{ for } x\in[a,b] \Rightarrow \int_a^b f(x)\,dx \leq 0$$

If $f(x)$ changes sign on $[a,b]$ then we can't say if the definite integral will be positive or negative without evaluating it.

Properties of Definite Integrals

1. $\displaystyle\int_a^a f(x)\,dx = 0$ (this is a definition)

2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\displaystyle\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$ (k = constant)

4. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

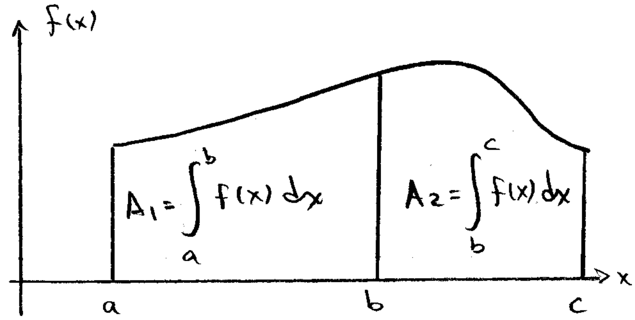
5. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. $\displaystyle\left\{\min_{x\in[a,b]} f(x)\right\}(b-a) \leq \int_a^b f(x)\,dx \leq \left\{\max_{x\in[a,b]} f(x)\right\}(b-a)$

7. If $f(x) \geq g(x)\ \forall\, x \in [a,b] \Rightarrow \displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

These all follow from the definition of the definite integral in terms of Riemann sums. See the book for proofs of the more interesting cases. Also, in interpreting these properties, remember that if $f(x) \geq 0$ on $[a,b]$ then $\int_a^b f(x)\,dx$ is the "area under the curve $f(x)$ on $[a,b]$". The area is negative if $f(x) \leq 0$ on $[a,b]$.

As one example:



The area under f(x) from a to c is clearly the area under f(x) from a to b plus the area from b to c, hence

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Example

(#2, sec 4.6)

Suppose that f and h are continuous and that

$$\int_1^9 f(x)\,dx = -1 \qquad \int_7^9 f(x)\,dx = 5 \qquad \int_7^9 h(x)\,dx = 4$$

Find: a) $\int_1^9 -2f(x)\,dx$

e) $\int_1^7 f(x)\,dx$

b) $\int_7^9 [f(x)+h(x)]\,dx$

f) $\int_9^7 [h(x)-f(x)]\,dx$

c) $\int_7^9 [2f(x)-3h(x)]\,dx$

d) $\int_9^1 f(x)\,dx$

Solution:

a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx$

$= (-2)(-1)$

$= 2$

b) $\int_7^9 [f(x) + h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx$

$= 5 + 4$

$= 9$

c) $\int_7^9 [2f(x) - 3h(x)]\,dx = \int_7^9 2f(x)\,dx + \int_7^9 (-3)h(x)\,dx$

$= 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx$

$= 2(5) - 3(4)$

$= 10 - 12$

$= -2$

d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx$

$= -(-1)$

$= 1$

e) $$\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx + \int_9^7 f(x)\,dx$$

$$= \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx$$

$$= (-1) - 5$$

$$= -6$$

f) $$\int_9^7 [h(x) - f(x)]\,dx = \int_9^7 h(x)\,dx - \int_9^7 f(x)\,dx$$

$$= -\int_7^9 h(x)\,dx + \int_7^9 f(x)\,dx$$

$$= -(4) + 5$$

$$= 1$$

Some Useful Definite Integrals

Using the sums on page 311 of the book it can be shown:

$$\boxed{\begin{aligned} \int_a^b dx &= b-a \\ \int_a^b x\,dx &= \frac{b^2-a^2}{2} \\ \int_a^b x^2\,dx &= \frac{b^3-a^3}{3} \\ \int_a^b x^3\,dx &= \frac{b^4-a^4}{4} \end{aligned}}$$

With these and the properties of definite integrals we can now evaluate several problems.

<u>Example</u>

(#14, sec 4.6)

$$\int_3^0 (2z-3)\,dz = -\int_0^3 (2z-3)\,dz$$

$$= -2\int_0^3 z\,dz + 3\int_0^3 dz$$

$$= -2\,\frac{(3)^2-(0)^2}{2} + 3\,(3-0)$$

$$= -9+9$$

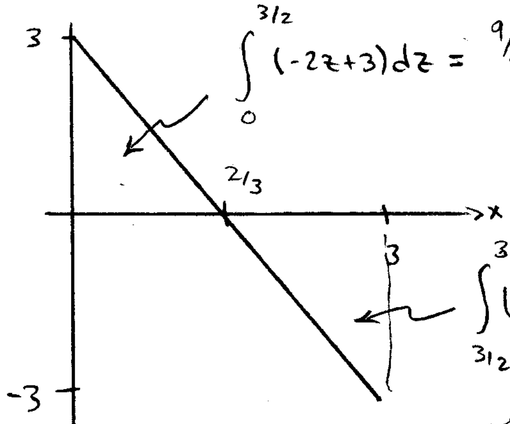
$$= 0$$

This is worth looking at graphically. Our integral is:

$$\int_3^0 (2z-3)\,dz = -\int_0^3 (2z-3)\,dz$$

$$= \int_0^3 (-2z+3)\,dz$$

$$= \int_0^{3/2} (-2z+3)\,dz + \int_{3/2}^3 (-2z+3)\,dz$$



$$\int_0^{3/2} (-2z+3)\,dz = 9/4 \quad \text{(pos. area)}$$

$$\int_{3/2}^{3} (-2z+3)\,dz = -9/4 \quad \text{(neg. area)}$$

<u>The areas are equal and opposite.</u>

Example
(#16, sec 4.6)

$$\int_{1/2}^{1} 24u^2\,du = 24\int_{1/2}^{1} u^2\,du$$

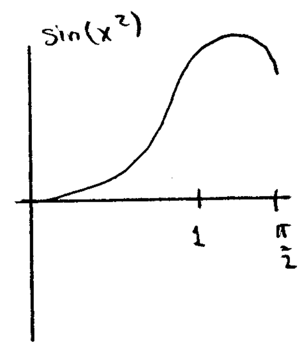
$$= 24\,\frac{(1)^3 - \left(\frac{1}{2}\right)^3}{3}$$

$$= 8\left(1 - \frac{1}{8}\right)$$

$$= 7$$

Example

Find minimum and maximum bounds on $\int_0^1 \sin(x^2)\,dx$



Solution:

$$\min_{x\in[0,1]} \sin(x^2) = 0$$

$$\max_{x\in[0,1]} \sin(x^2) = \sin(1)$$

From property 6 on page 2 of these notes:

$$0\,(1-0) \leq \int_0^1 \sin(x^2)\,dx \leq [\sin(1)][1-0]$$

$$\Rightarrow \quad 0 \leq \int_0^1 \sin(x^2)\,dx \leq \sin(1) = 0.84\ldots$$

The exact answer for the integral is $\int_0^1 \sin(x^2)\,dx = 0.3103\ldots$

Average Values of Continuous Functions

Recall that if we partition an interval $[a,b]$ into $a = x_0 < x_1 < \cdots < x_n = b$, then the approximate average of $f(x)$ on $[a,b]$ is:

$$\tilde{f}_n = \frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} = \frac{1}{n}\sum_{k=1}^{n} f(c_k) \qquad c_k \in [x_{k-1}, x_k]$$

Now suppose we choose equal length subintervals, so;

$$\Delta x_k = \Delta x = \frac{b-a}{n} \Rightarrow \frac{1}{n} = \frac{\Delta x}{b-a}$$

Then:

$$\tilde{f}_n = \frac{1}{n}\sum_{k=1}^{n} f(c_k)$$

$$= \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k)$$

$$= \frac{\sum_{k=1}^{n} f(c_k)\Delta x}{b-a}$$

Take the limit of this as $\|P\| \to 0$ and we get the exact average value of $f(x)$ over $b-a$.

$$\tilde{f} = \lim_{n\to\infty} \tilde{f}_n = \frac{\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x}{b-a} = \frac{\int_a^b f(x)\,dx}{b-a}$$

Hence:

Def If f is integrable on $[a,b]$, its <u>average</u> (<u>mean</u>) value on $[a,b]$ is

$$\tilde{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example
(#30, sec 4.6)
(1st part only)

Given $f(x) = 3x^2 - 3$ on $[0,1]$, find its mean value.

Solution:

$$\tilde{f} = \frac{1}{1-0}\int_0^1 (3x^2-3)\,dx$$

$$= 3\int_0^1 x^2 - 3\int_0^1 dx$$

$$= 3\left(\frac{1^3-0^3}{3}\right) - 3(1-0)$$

$$= 1 - 3$$

$$= -2$$

The Mean Value Theorem for Definite Integrals

Thm If f is continuous on $[a,b]$, then at some point $c \in [a,b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

This has an obvious geometrical interpretation. The definite integral is the area under $f(x)$ over $[a,b]$. Note that $f(c)(b-a)$ is the area under a constant function $f(c)$ over $[a,b]$. Then

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

just says that the constant function $f(c)$ has the same area as $f(x)$ does on $[a,b]$.

Example

(#43, sec 4.6) Suppose that f is continuous and that $\int_1^2 f(x)\,dx = 4$. Show that $f(x) = 4$ at least once on $[1,2]$.

Solution: For $a = 1$ and $b = 2$, from the MVT for definite integrals, there is some $c \in [a,b]$ where

$$f(c) = \frac{1}{2-1} \int_1^2 f(x)\,dx$$

$$= 4$$

$$\Rightarrow f(c) = 4$$

Hence there is at least one point in $[a,b]$ where $f(x) = 4$.