APPM 1350: Section 4.7: The Fundamental Theorem

Today we discuss the fundamental theorem of calculus. This theorem connects differential and integral calculus and represents on the most important (perhaps <u>the</u> most important) theorems in calculus.

Indefinite and Definite Integrals

So far we have introduced two types of integrals, the indefinite and the definite. The <u>indefinite integral takes a function and produces another function</u>

$$\underbrace{F(x)}_{\text{A function}} = \int \underbrace{f(x)}_{\text{A function}}\,dx$$

By contrast the <u>definite integral on a fixed interval $[a,b]$ takes a function and produces a number</u> (the area under the curve).

$$\underbrace{\text{"Area under } f(x)\text{"}}_{\text{A number}} = \int_a^b \underbrace{f(x)}_{\text{A function}}\,dx$$

But suppose we let one of the limits of integration be a variable, in particular

$$\int_a^x f(x)\,dx$$

or equivalently

$$\int_a^x f(y)\,dy$$

where we use y as the dummy variable in the integral. This is still a definite integral, <u>but this time the definite integral is a function!</u>

$$F(x) = \int_a^x f(y)\,dy$$

Geometrically, F(x) is the area under the curve f(x) on the interval [a, x]. It is a function because we let the interval be a function of x. So, for example, consider

$$F(x) = \int_2^x y^2\,dy = \frac{x^3 - 2^3}{3}$$

This is the area under y^2 over [2, x].

The Fundamental Theorem of Calculus

The relation

$$F(x) = \int_a^x f(y)\,dy \qquad (a = \text{some constant})$$

relates a function F(x) to any integrable function f(x). What else can we say about this connection? In particular, what can we say about the derivative of F(x) and the function f(x)? Let's just explore it a bit. We know

$$\frac{d}{dx}F(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

So let's construct the difference quotient using our definition of F(x) as a definite integral and the properties of definite integrals.

So; assume $f(x)$ is continuous on $[a,b]$. Then for $x \in [a,b]$;

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left\{ \int_a^{x+h} f(y)\,dy - \int_a^{x} f(y)\,dy \right\}$$

$$= \frac{1}{h}\left\{ \int_a^{x} f(y)\,dy + \int_x^{x+h} f(y)\,dy - \int_a^{x} f(y)\,dy \right\}$$

$$= \frac{1}{h}\int_x^{x+h} f(y)\,dy$$

$$= f(c) \qquad c \in [x,x+h]$$

where the last step follows from the mean value theorem of definite integrals (note that $\frac{1}{h}\int_x^{x+h} f(x)\,dx$ is the average of $f(x)$ on $[x,x+h]$). Thus we have

$$\frac{d}{dx}F(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(c) = f(x)$$

where the limit $\lim_{h\to 0} f(c) = f(x)$ follows from the fact that the interval

$[x,x+h]$ is squeesing into just the point x. Hence a point c in $[x,x+h]$ is approaching x as $h \to 0$. We have just proven:

> **Theorem:** The Fundemental Theorem of Calculus, Part 1
>
> If f is continuous on $[a,b]$, then
>
> $$F(x) = \int_a^x f(y)\,dy$$
>
> has a derivative at every point of $[a,b]$ and
>
> $$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(y)\,dy = f(x) \qquad x \in [a,b]$$

Super important! Remember!

Condensing our result, we just showed that

$$\frac{dF}{dx} = f$$

has a solution for every continuous function $f(x)$. The solution $F(x)$ is the definite integral

$$F(x) = \int_a^x f(y)\,dy$$

where a can be any value (if there is an initial condition associated with the differential equation, a is determined by that initial condition).

Geometrically, the result

$$\frac{d}{dx}\int_a^x f(y)\,dy = f(x)$$

says that the rate at which the area under $f(x)$ is changing is just the value of the function itself (see figure 4.23 on page 334 of the book).

Example

$$\frac{d}{dx}\int_0^x y^3\,dy = x^3$$

Example

$$\frac{d}{dx}\int_{-100}^x \frac{\sqrt{\cos^3(y)+y^5}}{y^4+1}\,dy = \frac{\sqrt{\cos^3(x)+x^5}}{y^4+1}$$

Example

(#59, sec 4.7) Express the solution of the IVP

$$\frac{dy}{dx} = \sec x \qquad y(2) = 3$$

in terms of integrals.

Solution:

$\sec x$ is continuous on the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, so on some closed interval $\left[\frac{\pi}{2}+\epsilon, \frac{3\pi}{2}-\epsilon\right]$ where $\epsilon > 0$ and $\epsilon \ll 1$. The initial point $x=2$ is in this interval, Then from the fundamental theorem of calculus we know the differential equation has a solution in this intervel and further;

$$y(x) = \int_a^x \sec y \, dy$$

where a is in our interval. To eliminate the constant a, use the initial condition

$$y(2) = 3 = \int_a^2 \sec y \, dy$$

Then assuming $a < 2$, we have:

$$\begin{aligned} y(x) &= \int_a^x \sec y \, dy \\ &= \int_a^2 \sec y \, dy + \int_2^x \sec y \, dy \\ &= 3 + \int_2^x \sec y \, dy \end{aligned}$$

Evaluating Definite Integrals

We now explore the second part of the fundamental theorem of calculus, the part that tells us how to evaluate definite integrals in terms of indefinite integrals.

The first part of the fundamental theorem says that if

$$G(x) = \int_a^x f(y)\,dy \qquad (1)$$

for $f(x)$ continuous on $[a,b]$, then

$$\frac{d}{dx} G(x) = f(x) \qquad (2)$$

So $G(x)$ defined as a definite integral of $f(x)$ is in fact an <u>antiderivative</u> of $f(x)$. Now, integrating (2) as an <u>indefinite integral</u>, we get

$$G(x) = \int f(x)\,dx + c$$

$$= F(x) + c \qquad (3)$$

where $F(x)$ is the nonconstant part of the indefinite integral. Equating (1) and (3) we have

$$\int_a^x f(y)\,dy = \int f(x) + c = F(x) + c$$

$$\Rightarrow F(x) = \int_a^x f(y)\,dy - c$$

So:

$$F(b) - F(a) = \int_a^b f(y)\,dy - c - \cancelto{0}{\int_a^a f(y)\,dy} + c = \int_a^b f(y)\,dy$$

So to evaluate the definite integral

$$\int_a^b f(y)\,dy$$

we compute the indefinite integral of $f(x)$:

$$F(x) = \int f(x)$$

and subtract it's value when evaluated at each of the limits of integration:

$$\int_a^b f(y)\,dy = F(b) - F(a)$$

This is the second part of the fundamental theorem.

Theorem: The Fundamental Theorem of Calculus, Part 2

If $f(x)$ is continuous at every point of $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ on $[a,b]$, then

$$\int_a^b f(y)\,dy = F(b) - F(a)$$

Super important! Remember!

So we have shown how definite integrals are related to indefinite integrals and (using part 1 of the fundamental theorem) how derivatives of definite integrals are related to the integrand of the integral. We now have a complete connection between integrals and derivatives. We also now have a way of evaluating definite integrals without having to compute Riemann sums.

Example
(#12, sec 4.7) Evaluate $$\int_{\pi/6}^{5\pi/6} \csc^2 x \, dx$$

Solution:

$$F(x) = \int \csc^2 x \, dx = -\cot x + c$$

and from the fundamental theorem of calculus; noting $\csc^2 x$ is cont. on $[\frac{\pi}{6}, \frac{5\pi}{6}]$;

$$\int_{\pi/6}^{5\pi/6} \csc^2 x \, dx = \left[-\cot\left(\tfrac{5\pi}{6}\right) + c\right] - \left[-\cot\left(\tfrac{\pi}{6}\right) + c\right]$$

$$= -\cot\left(\tfrac{5\pi}{6}\right) + \cot\left(\tfrac{\pi}{6}\right)$$

$$= -(-\sqrt{3}) + \sqrt{3}$$

$$= 2\sqrt{3}$$

Example
(#18, sec 4.7) Evaluate $$\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$$

Solution: The integrand is continuous on $[-\frac{\pi}{3}, -\frac{\pi}{4}]$ (it is essential to insure that). Then

$$F(x) = \int \left(4\sec^2 x + \frac{\pi}{x^2}\right) dx$$

$$= 4\tan x - \frac{\pi}{x} + c$$

For the application of the fundamental theorem we can use any antiderivative, so we can set $c = 0$ and applying the fundamental theorem:

$$\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt = F\left(-\frac{\pi}{4}\right) - F\left(-\frac{\pi}{3}\right)$$

$$= \left[4\tan\left(-\frac{\pi}{4}\right) - \frac{\pi}{\left(-\frac{\pi}{4}\right)}\right] - \left[4\tan\left(-\frac{\pi}{3}\right) - \frac{\pi}{\left(-\frac{\pi}{3}\right)}\right]$$

$$= \left[-4\tan\frac{\pi}{4} + 4\right] - \left[-4\tan\frac{\pi}{3} + 3\right]$$

$$= \left[-4 + 4\right] - \left[-4\sqrt{3} + 3\right]$$

$$= 4\sqrt{3} - 3$$

<u>A very important point!</u>

To apply the fundamental theorem to calculate

$$\int_a^b f(x)\,dx$$

it is <u>absolutely essential that $f(x)$ be continuous on $[a,b]$.</u> If it is not then you can't apply the fundamental theorem to evaluate the integral. For example;

$$F(x) = \int \frac{dx}{x^2} = -\frac{1}{x} + C$$

But

$$\int_{-1}^{1} \frac{dx}{x^2} \neq F(1) - F(-1) = -2$$

because $\frac{1}{x^2}$ is discontinuous at $x=0$. In point of fact the area under this function over $[-1,1]$ is infinite, not -2! <u>Failing to check for continuity is one of the most common errors made when evaluating definite integrals.</u>

More Examples of Evaluating Definite Integrals

Example

(#36, sec 4.7) Find the total area between the region $y = 3x^2 - 3$ and the x-axis for $-2 \leq x \leq 2$.

Solution:

In this case we define the total area between the curve and the x axis as being the absolute value of the area, or put differently, as the definite integral of the absolute value (this is because $\int_a^b f(x)dx < 0$ if $f(x) < 0$ on $[a, b]$). Hence

$$\text{Area} = \int_{-2}^{2} |3x^2 - 3|\, dx$$

Now, $3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$ and

$$3x^2 - 3 \quad \begin{cases} \geq 0 & \text{for } [-2, -1] \\ \leq 0 & \text{for } [-1, 1] \\ \geq 0 & \text{for } [1, 2] \end{cases}$$

So:

$$\text{Area} = \int_{-2}^{2} |3x^2 - 3|\, dx = \int_{-2}^{-1} (3x^2 - 3)\,dx \underbrace{- \int_{-1}^{1} (3x^2 - 3)\,dx}_{\text{using } |x| = -x \text{ if } x \leq 0} + \int_{1}^{2} (3x^2 - 3)\,dx$$

Since

$$F(x) = \int (3x^2 - 3)\,dx = x^3 - 3x + C$$

we have

$$\begin{aligned} \text{Area} &= [F(-1) - F(-2)] - [F(1) - F(-1)] + [F(2) - F(1)] \\ &= 4 + 4 + 4 = 12 \end{aligned}$$

Example
(#52, sec 4.7) Find $\frac{dy}{dx}$ given $y = \int_0^{x^2} \cos\sqrt{t}\, dt$

Solution:

$y(x)$ is a composite function;

$$y(x) = g(x^2)$$

where

$$g(x) = \int_0^{x} \cos\sqrt{t}\, dt$$

So from the chain rule:

$$\frac{d}{dx} y(x) = \underbrace{g'(x^2)}_{\substack{g'(x) \text{ evaluated} \\ \text{at } x^2}} \frac{d}{dx} x^2$$

$$= (\cos\sqrt{x^2})(2x) \qquad \text{since } \frac{d}{dx}\int_0^x \cos\sqrt{t}\, dt = \cos\sqrt{x}$$

$$= 2x\cos x$$

Example
(#53, sec 4.7) Find $\frac{dy}{dx}$ given $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}$ $\quad |x| < \frac{\pi}{2}$

Solution: Note that the restriction $|x| < \pi/2 \Rightarrow$ the integrand is continuous which means we can apply the fundamental theorem. So:

$$\frac{d}{dx}\int_0^{f(x)} \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\sqrt{1-[f(x)]^2}} \frac{d}{dx} f(x)$$

In our case $f(x) = \sin x$, hence:

$$\frac{d}{dx}\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\sqrt{1-\sin^2 x}} \cos x = \frac{\cos x}{\sqrt{\cos^2 x}} = 1 \quad \text{for } |x| < \frac{\pi}{2}$$

Example

Find the equation of the tangent line to the curve $y(x) = F(x)$ where

$$F(x) = \int_x^1 \sqrt[3]{t^2+7}\, dt$$

at the point on the curve where $x = 1$.

Solution:

$$F(x) = \int_x^1 (t^2+7)^{1/3}\, dt$$

$$= -\int_1^x (t^2+7)^{1/3}\, dt$$

Then $F(1) = 0$ (upper and lower limits of integration are the same) and

$$F'(x) = -(x^2+7)^{1/3}$$

$$\Rightarrow F'(1) = -(1+7)^{1/3}$$

$$= -2$$

Hence the tangent line to $F(x)$ at $x = 1$ is:

$$y_t(x) = F(1) + F'(1)(x-1)$$

$$= 0 - 2(x-1)$$

$$= 2 - 2x$$