APPM 1350: Section 6.2: Natural Logarithms

The logarithm was one of the most important inventions, or if you prefer discoveries, of all times. Logarithms make multiplication and division into addition and subtraction. This greatly simplified calculation and made possible slide rules and other clever calculating devices. Logarithms are still used for some calculations done by digital computers, in particular evaluating powers of numbers. Finally they are an important function in and of themselves that occurs frequently, particularly in science and engineering. The natural logarithm is defined as a definite integral:

Def: The Natural Logarithm

$$\ln x = \int_1^x \frac{dt}{t} \qquad (x > 0)$$

Note the restriction $x > 0$! Logarithms are only defined for positive numbers.

Note also;

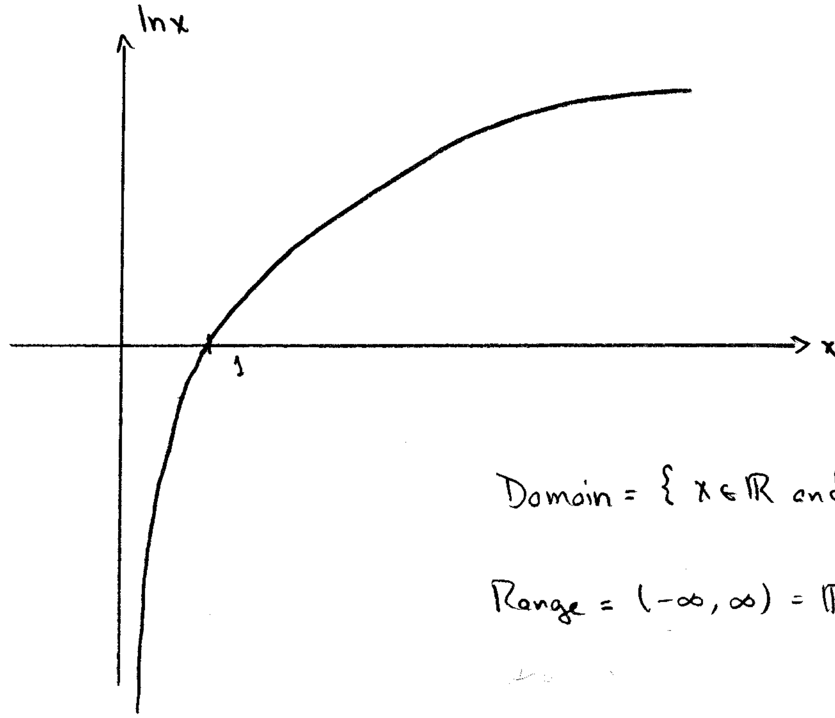
$$\ln 1 = \int_1^1 \frac{dt}{t} = 0$$

since the upper and lower limits of integration are the same. For $0 < x < 1$, we also have:

$$\ln x = \int_1^x \frac{dt}{t} = -\int_x^1 \frac{dt}{t}$$

Since $\int_x^1 \frac{dt}{t} > 0$ for $0 < x < 1$ we find that $\ln x < 0$ for $0 < x < 1$. By the same token; $\ln x > 0$ for $x > 1$. Finally, as $x \to 0$ then $\ln x \to -\infty$ because the area under the curve becomes infinite. Similarly as $x \to \infty$ then $\ln x \to \infty$.

Graphically:



Domain = $\{ x \in \mathbb{R} \text{ and } x > 0 \}$

Range = $(-\infty, \infty) = \mathbb{R}$

The Derivative of $y = \ln x$

From the fundamental theorem of calculus;

$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x \frac{dt}{t} = \frac{1}{x}$$

So:

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$$

and from the chain rule; if $u = u(x)$ (i.e., u is a function of x), then

$$\boxed{\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx} \qquad u(x) > 0}$$

Example

(# 8, sec 6.2)

$$\frac{d}{dt}\ln(t^{3/2}) = \frac{1}{t^{3/2}}\frac{d}{dt}t^{3/2}$$

$$= \frac{1}{t^{3/2}}\left(\frac{3}{2}t^{1/2}\right)$$

$$= \frac{3}{2t}$$

Example

(# 21, sec 6.2)

$$\frac{d}{dx}\frac{\ln x}{1+\ln x} = \frac{(1+\ln x)\frac{d}{dx}\ln x - \ln x\frac{d}{dx}(1+\ln x)}{(1+\ln x)^2}$$

$$= \frac{(1+\ln x)\frac{1}{x} - \ln x\left(\frac{1}{x}\right)}{(1+\ln x)^2}$$

$$= \frac{1}{x(1+\ln x)^2}$$

Example

(# 35, sec 6.2)

$$\frac{d}{dx}\int_{x^2/2}^{x^2}\ln\sqrt{t}\,dt = \frac{d}{dx}\left[\int_{x^2/2}^{a}\ln\sqrt{t}\,dt + \int_{a}^{x^2}\ln\sqrt{t}\,dt\right] \quad (a>0)$$

$$= -\frac{d}{dx}\int_{a}^{x^2/2}\ln\sqrt{t}\,dt + \frac{d}{dx}\int_{a}^{x^2}\ln\sqrt{t}\,dt$$

$$= -\left(\ln\sqrt{\frac{x^2}{2}}\cdot x\right) + \left(\ln\sqrt{x^2}\cdot 2x\right)$$

$$= 2x\ln|x| - x\ln\frac{|x|}{\sqrt{2}}$$

Properties of Logarithms

For any numbers $a > 0$ and $x > 0$

1. $\ln ax = \ln a + \ln x$

2. $\ln \frac{a}{x} = \ln a - \ln x$

3. $\ln \frac{1}{x} = -\ln x$

4. $\ln x^n = n \ln x \qquad (n \in \mathbb{R})$

The proof of #1 is cute and worth doing.

Thm: $\ln ax = \ln a + \ln x$ for $a > 0$ and $x > 0$

Proof:

Note that:

$$\frac{d}{dx} \ln ax = \frac{1}{ax} a = \frac{1}{x}$$

So: $\ln x$ and $\ln ax$ have the same derivative, hence both $\ln x$ and $\ln ax$ are antiderivatives of the same function $\left(\frac{1}{x}\right)$ and therefore differ by at most a constant. Hence

$$\ln ax = \ln x + c$$

Since this holds for any $x > 0$, try $x = 1$. Since $\ln 1 = 0$, we have

$$\ln a = c$$

$$\Rightarrow \quad \ln ax = \ln x + \ln a$$

$\blacksquare$

Proofs of the other rules are in Thomas and are worth a look.

Logarithm Differentiation

Sometimes derivatives are easier to compute if you first take the logarithm. This is called <u>logarithmic differentiation</u>.

Example

(#38, sec 6.2) Compute dy/dx given $y = \sqrt{(x^2+1)(x-1)^2}$

Solution:

Take the logarithm of both sides and simplify.

$$\ln y = \ln \sqrt{(x^2+1)(x-1)^2}$$

$$= \ln (x^2+1)^{1/2} |x-1|$$

$$= \frac{1}{2} \ln (x^2+1) + \ln |x-1|$$

Now implicitly differentiate:

$$\frac{d}{dx} \ln y = \frac{1}{2} \frac{d}{dx} \ln (x^2+1) + \frac{d}{dx} \ln (x-1)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{2x}{2(x^2+1)} + \frac{1}{(x-1)}$$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{x}{x^2+1} + \frac{1}{(x-1)} \right)$$

$$= y \left(\frac{x(x-1) + x^2+1}{(x^2+1)(x-1)} \right)$$

$$= y \left(\frac{x^2 - x + x^2+1}{(x^2+1)(x-1)} \right)$$

$$= \sqrt{(x^2+1)(x-1)^2} \quad \frac{2x^2 - x + 1}{(x^2+1)(x-1)}$$

Substituting the original expression for y

$$= \frac{(2x^2 - x + 1)|x-1|}{\sqrt{x^2+1}\ (x-1)}$$

This is not a great example of logarithmic differentiation being simplier than straight differentiation, but occasionally it really is.

Example

Compute $\frac{dy}{dx}$ given $y = \sqrt[3]{\frac{x+2}{x-1}}$.

Solution:

$$\ln y = \ln \sqrt[3]{\frac{x+2}{x-1}}$$

$$= \frac{1}{3} \ln \frac{x+2}{x-1}$$

$$= \frac{1}{3}\left[\ln(x+2) - \ln(x-1) \right]$$

$$\Rightarrow \frac{d}{dx} \ln y = \frac{1}{3}\left[\frac{d}{dx} \ln(x+2) - \frac{d}{dx} \ln(x-1) \right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{3(x+2)} - \frac{1}{3(x-1)}$$

$$= \frac{1}{3} \frac{(\cancel{x}-1) - (\cancel{x}+2)}{(x-1)(x+2)}$$

$$= -\frac{1}{(x-1)(x+2)}$$

$$\Rightarrow \frac{dy}{dx} = y\left[-\frac{1}{(x-1)(x+2)}\right]$$

$$= \frac{(x+2)^{1/3}}{(x-1)^{1/3}} \cdot \left[-\frac{1}{(x-1)(x+2)}\right]$$

$$= -\frac{1}{(x-1)^{4/3}(x+2)^{2/3}}$$

The Integral $\int \frac{du}{u}$

If $u = u(x)$, we noted earlier that

$$\frac{d}{dx}\ln u(x) = \frac{1}{u(x)}\frac{du}{dx} \qquad (u>0)$$

So

$$d\ln u = \frac{1}{u}\,du$$

$$\Rightarrow \int d\ln u = \int \frac{1}{u}\,du$$

$$\Rightarrow \int \frac{du}{u} = \ln u + c$$

provided $u>0$. If $u<0$, we note that $-u>0$, in which case

$$\int \frac{du}{u} = \int \frac{dg}{g}$$ where $g = -u \Rightarrow dg = -du$ Both g and dg are positi

$$= \ln g + c$$

$$= \ln(-u) + c = \ln|u| + c$$

Hence

> If u is a nonzero differentroble function, then
>
> $$\int \frac{du}{u} = \ln|u| + C$$

Recall the power rule for integration

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

Now we know how to handle the $n=-1$ case.

Example

(#52, sec 6.2)

$$\int_{-1}^{0} \frac{3\,dx}{3x-2} = \int_{-5}^{-2} \frac{du}{u}$$

Let $u = 3x-2$
$du = 3dx$

$$= \ln|u| \Big|_{-5}^{-2}$$

$$= \ln 2 - \ln 5$$

$$= \ln\left(\frac{2}{5}\right)$$

<u>Example</u>
(#56, sec 6.2)

$$\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta = \int_{-3}^{-1} \frac{du}{u}$$

Let $u = 1-4\cos\theta$
$du = 4\sin\theta\, d\theta$

$$= \ln|u| \Big|_{-3}^{-1}$$

$$= \ln 1 - \ln 3$$

$$= -\ln 3$$

$$= \ln \frac{1}{3}$$

The Integrals of tan x and cot x

We now have the machinery to calculate the integrals of $\tan x$ and $\cot x$. Thus:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

$$= -\int \frac{du}{u}$$

Let $u = \cos x$
$\Rightarrow du = -\sin x\, dx$

$$= -\ln|u| + c$$

$$= -\ln|\cos x| + c$$

$$= \ln|\sec x| + c$$

and similiarly;

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$$

Let $u = \sin x$
$du = \cos x\,dx$

$$= \int \frac{du}{u}$$

$$= \ln|u| + c$$

$$= \ln|\sin x| + c$$

$$= -\ln|\csc x| + c$$

So:

$$\boxed{\begin{aligned} \int \tan x\,dx &= -\ln|\cos x| + c = \ln|\sec x| + c \\ \int \cot x\,dx &= \ln|\sin x| + c = -\ln|\csc x| + c \end{aligned}}$$

Example
(#66, sec 6.2)

$$\int_0^{\pi/12} 6\tan 3x\,dx = 2\int_0^{\pi/4} \tan u\,du$$

Let $u = 3x$
$du = 3\,dx$

$$= 2\ln|\sec u|\Big|_0^{\pi/4}$$

$$= 2\left(\ln\sqrt{2} - \ln 1\right)$$

$$= 2\ln\sqrt{2}$$

$$= \ln 2$$

A Final Integration Example

This puppy is kind of nasty, but interesting.

Example
(#68, sec 6.2)

$$\int \frac{\sec x \, dx}{\sqrt{\ln(\sec x + \tan x)}} = \int \frac{du}{u\sqrt{\ln u}}$$

Let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\, dx$

$= \sec x(\tan x + \sec x)\, dx$

$= u \sec x \, dx$

$$= \int \frac{dw}{w^{1/2}}$$

Let $w = \ln u$

$dw = \frac{1}{u}\, du$

$$= 2\sqrt{w} + c$$

$$= 2\sqrt{\ln u} + c$$

$$= 2\sqrt{\ln(\sec x + \tan x)} + c$$

Sweet!